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Nonlinear heat diffusion simulation using Volterra series expansion

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A method for solving a nonlinear heat diffusion problem based on the use of the Volterra series is presented. Application in a practical configuration shows that the method comes to solve a linear problem at time t with a source term that depends on solutions calculated at previous instants. Unknowns of this linear problem are the generalized transfer functions $H_k(p_1,\ldots p_k)$, $k = 1,2,\ldots$, that are the kth order Laplace transforms $\mathscr{L}^{(k)}$ of the Volterra kernels $h_k(t)$. The method allows splitting naturally the solution as a linear contribution and a nonlinear one. Interest of such a method stands on the fact that a very small number of kernels is required to simulate accurately the nonlinear contribution (in practice 1 or 2). Furthermore, it is demonstrated that the Volterra method can be used efficiently to find the expression of a single simplified kernel that simulates well the entire nonlinear contribution.

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1. Introduction

This study aims at presenting the Volterra series as a mathematical tool to solve nonlinear heat transfer problems. Although there is no restriction concerning the heat transfer mode, the present paper focuses on heat transfer by diffusion. The nonlinearity is due to the temperature dependence of thermal conductivity and specific heat per unit volume.

The Italian mathematician Vito Volterra first introduced the notion of what is now known as a Volterra series in his “Theory of Functionals” (see Refs. [1] and [2] for the mathematical aspects and [9] for a more accessible presentation for the physicist). A nice application to nonlinear acoustics has been presented rather recently in Ref. [3]. Such application highlights the efficacy of the method in terms of parametric parsimony to simulate separately either the linear or the nonlinear system behaviors. Consequently, this method allows calculating accurately the nonlinear contribution.

Let us illustrate the concept considering a monovariable thermal system, with a heat flux $\varphi(t)$ as system input and a temperature $T(t)$ at one point inside the system as system output. If the system behaves as a linear causal one with memory, we know that the temperature at time $t = n\Delta t$ depends on the contribution at time t and at previous instants until $t = 0$. Therefore, the output of a linear dynamical system is expressed as the sum of weighted contributions of the past effects:

$$T(n\Delta t) = \sum_{i=0}^{n} h_1(i\Delta t)\varphi((n-i)\Delta t)\Delta t \quad (1)$$

Considering that $\Delta t \to 0$ comes to replace the previous discrete expression as a continuous-time one. The classical convolution is hence obtained:

$$T(t) = h_1(t)\star\varphi(t) = \int_0^t h_1(\tau)\varphi(t-\tau)\mathrm{d}\tau \quad (2)$$

where $h_1(t)$ denotes the impulse response of the linear system, that is the response to the heat flux generated as a Dirac function: $\varphi(t) = \delta(t)$. This is at the basis of the classical Duhamel theorem presented in Ref. [4].

If the system behaves non-linearly around $t = t_i$, the temperature and the heat flux can be locally related using a series expansion such as:

$$T(t_i) = \gamma_1^i\varphi(t_i) + \gamma_2^i\varphi(t_i)^2 + \ldots = \sum_{j=1}^{\infty}\gamma_j^i[\varphi(t_i)]^j \quad (3)$$

In this relation, parameters γ_j^i of the polynomial are obtained at the time instant t_i. As represented in Fig. 1, the relation (3) comes to fit the nonlinear function $T(t_i)$ at time t_i using enough terms in the series. The main idea of Volterra was to introduce the memory

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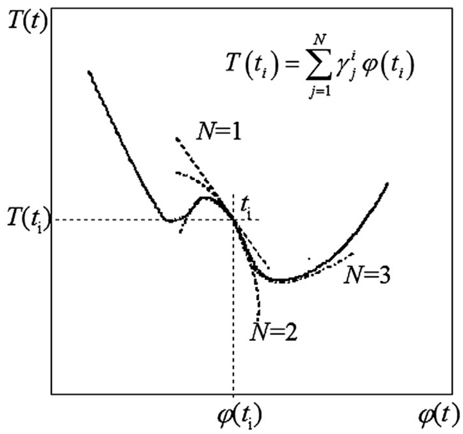

Fig. 1. Approaching the temperature $T(t)$ from a series expansion of the heat flux $\varphi(t)$ at $t = t_i$.

effect in this series expansion which leads to express $T(t)$ by weighted past values of $\varphi(t)$ when t starts at the initial time as:

$$
\begin{aligned}
T(t) &= \int_0^\infty h_1(\tau)\varphi(t-\tau)\mathrm{d}\tau \\
&\quad + \int_0^\infty\int_0^\infty h_2(\tau_1,\tau_2)\varphi(t-\tau_1)\varphi(t-\tau_2)\mathrm{d}\tau_1\mathrm{d}\tau_2 + \ldots \\
&= \sum_{j=1}^{\infty}\int_0^\infty\ldots\int_0^\infty h_j(\tau_1,\ldots,\tau_j)\prod_{i=1}^{j}\varphi(t-\tau_i)\mathrm{d}\tau_i
\end{aligned}
\tag{4}
$$

This result can be represented as an infinite sum of contributions:

$$
T(t) = \sum_{j=1}^{\infty}T_j(t) = T_1(t)+T_2(t)+\ldots+T_j(t)+\ldots \tag{5}
$$

with:

$$
T_j(t) = \int_0^\infty\ldots\int_0^\infty h_j(\tau_1,\ldots,\tau_j)\prod_{i=1}^{j}\varphi(t-\tau_i)\mathrm{d}\tau_i \tag{6}
$$

where $h_j(\tau_1,\ldots,\tau_j)$ is called the jth order Volterra kernel. Considering $j = 1$ comes to the linear contribution expressed initially in (2) which means that the first order kernel is related only to the linear contribution. However, since this approach is based on the local approximation (3), the Volterra method remains theoretically efficient for weak non-linearities. This limitation must be understood as the respective contributions of the linear ($n = 1$) and nonlinear ($n = 2,\ldots,\infty$) terms in the sum (5). In other words, the linear contribution must remain dominant in the system output.

The jth order Laplace transform $\mathscr{L}^{(j)}$ of the kernel $h_j(t_1,\ldots,t_j)$ is:

$$
H_j\left(p_1,\ldots,p_j\right) = \int_0^\infty\ldots\int_0^\infty h_j(\tau_1,\ldots,\tau_j)e^{-p_1\tau_1-\ldots-p_j\tau_j}\mathrm{d}\tau_1\ldots\mathrm{d}\tau_j \tag{7}
$$

Usually, $H_j(p_1,\ldots,p_j)$ is called the jth generalized transfer function. Applying this transform to (6) yields (see the Appendix):

$$
\theta_j\left(p_1,\ldots,p_j\right) = H_j\left(p_1,\ldots,p_j\right)\prod_{i=1}^{j}\psi(p_i) \tag{8}
$$

where $\psi(p)$ is the $\mathscr{L}^{(1)}$ Laplace Transform of $\varphi(t)$. It is therefore possible to express $T_j(t)$ as the jth order inverse Laplace transform of $\theta_j(p_1,\ldots,p_j)$ as:

$$
\begin{aligned}
T_j(t) &= \mathscr{L}^{(j)^{-1}}\left[\theta_j\left(p_1,\ldots,p_j\right)\right] \\
&= \frac{1}{(2i\pi)^n}\int_{c-i\infty}^{c+i\infty}\ldots\int_{c-i\infty}^{c+i\infty}\theta_j\left(p_1,\ldots,p_j\right)e^{p_1t}\ldots e^{p_jt}\mathrm{d}p_1\ldots\mathrm{d}p_j
\end{aligned}
\tag{9}
$$

where c is a real number chosen within the convergence domain of $\theta_j(p_1,\ldots,p_j)$. In this paper, the $\mathscr{L}^{(k)^{-1}}$ inverse Laplace transform is calculated using the Gaver-Stehfest numerical algorithm described in Refs. [7]; however, other techniques are available as proposed in Ref. [8].

From a mathematical point of view, the convergence of the series (5) holds for inputs $\varphi(t)$ such that $\varphi(t) < \varphi_0$ where φ_0 is viewed as a convergence radius that is determined from the characteristic function (3) where $\gamma_j = \int_{\mathbb{R}^n}h_j(\tau_1,\ldots,\tau_j)\mathrm{d}\tau_1\ldots\mathrm{d}\tau_j$ [6]. However, we will not study the convergence of the Volterra series in this paper since, as presented in the following, very low-order truncations of the series will yield to good approximations.

2. Application to the non-linear heat diffusion

2.1. From 1D heat equation to Volterra kernels

Consider a semi-infinite medium of which the surface is submitted to a time dependent uniform heat flux $\varphi(t)$. The specific heat per unit volume and the thermal conductivity are assumed to vary linearly with the temperature $T(t)$ as:

$$
\rho C_p(T) = C_0 + C_1T \tag{10}
$$

$$
k(T) = k_0 + k_1T \tag{11}
$$

where C_0, C_1, k_0 and k_1 are constant and T is in °C. The one-dimensional heat transfer model in the medium is mathematically described by the following partial differential equation:

$$
\rho C_p(T(x,t))\frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(k(T(x,t))\frac{\partial T(x,t)}{\partial x}\right),\quad 0<x<\infty,\quad t>0 \tag{12}
$$

with associated boundary conditions:

$$
-k(T(x,t))\frac{\partial T(x,t)}{\partial x} = \varphi(t),\quad x = 0,\quad t>0 \tag{13}
$$

$$
T(x,t) = 0,\quad x\to\infty,\quad t>0 \tag{14}
$$

and the initial condition:

$$
T(x,t) = 0,\quad 0\le x<\infty,\quad t = 0 \tag{15}
$$

Using the expression of $k(T)$ and $\rho\,C_p(T)$ in (10 and 11), replacing $T(x,t)$ in (12) with the series given by (5) and identifying each order yield:

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