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Free convection stagnation-point boundary-layer flow in a porous medium with a density maximum

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ABSTRACT

The steady free convection boundary-layer flow near a stagnation point in a fluid-saturated porous medium is considered when the convecting fluid is close to its maximum density. Three forms for the wall boundary condition are treated, a prescribed wall temperature, prescribed wall heat flux and Newtonian heating. In each case the flow and heat transfer characteristics are determined by a dimensionless parameter δ that measures the difference between the ambient temperature and the temperature at which the fluid attains its density maximum. We find that solutions are possible for $\delta \geq 0$ for each case. For $\delta < 0$ there is a critical value δ_c of δ, the value of which depends on the boundary conditions applied, with solutions possible only for $\delta \geq \delta_c$. The nature of this critical value, as well as other limiting asymptotic forms is discussed.

1. Introduction

Free convection boundary-layer flows arising from density gradients within a fluid-saturated porous medium in a gravitational field are the subject of much ongoing research in the more general area of fluid mechanics and heat transfer because of their important applications in environmental, geophysical and energy related engineering problems, see [1–3] for examples. To analyse these flows the Boussinesq approximation [2] is commonly used together with a linear density–temperature relation. Water, as well as several metals, has its maximum density in the liquid phase. For example, pure water at atmospheric pressure attains its density maximum at about a temperature of $T_m = 3.98^o$ C and this density reversal for lower temperatures can have significant effects on any buoyancy-driven flow. Goren [4] proposed a new relation in which the density difference varies with the square of the temperature difference for those cases where the usual linear density–temperature relation is not adequate. Gebhart and Mollendorf [5] developed a more accurate density relationship for water around the density extremum condition for different salinity levels. Using this relation [6] they were able to find similarity solutions for two-dimensional boundary-layer flows induced by the buoyancy effects of thermal and saline diffusion.

Kay *et al.* [7] analysed the thermal bar, a descending planar plume of denser fluid at temperature T_m in a lighter fluid at temperatures above or below T_m, as a laminar free convection boundary-layer flow, using the density relation proposed by Goren [4]. More recently Cayley and McBride [8] studied the free convection flow in a vertical cylinder of water in the vicinity of the density maximum at about 4 °C both experimentally and theoretically using several density–temperature relations. One of their density–temperature relations, and the one that we shall consider in this paper, has a parabolic variation in temperature T and is claimed [8] to be valid for the range of temperatures from 0 °C to 100 °C. In [8] they also present experimental evidence for the formation of a rising vortex of water, starting in the lower edge regions of the cylinder.

In this paper we consider how a density maximum can affect free convection boundary-layer flows within a fluid-saturated porous medium. We model the flow in the porous medium by Darcy's law and take the density–temperature relation suggested by Cayley and McBride [8]. We also restrict our attention to the flow near a lower stagnation point. This simplification of the flow geometry allows the steady problem to be reduced to similarity form, which we can then treat in detail thus enabling the specific features of a density reversal to be clearly brought out. We treat three separate types of boundary condition, namely a prescribed wall temperature, a prescribed wall heat flux and Newtonian

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Nomenclature

c_1, c_2	constants in the density–temperature relation (1).
$f(y)$	(dimensionless) form for the streamfunction, defined in (18).
g	acceleration due to gravity.
h_s	wall heat transfer coefficient.
K	permeability of the porous medium.
l	length scale.
q_w	wall heat flux.
Ra	Rayleigh number.
T	fluid temperature.
T_m	temperature at density maximum.
T_w	wall temperature.
T_∞	ambient temperature.
T_s	temperature scale, Eqs. (10,11).
$\bar{u}, \bar{v}$	velocity components in the $\bar{x}$ and $\bar{y}$ directions.
U_0	velocity scale, Eqs. (10,11).
$\bar{x}, \bar{y}$	co-ordinates along and normal to the surface.
Greek symbols	
α	effective thermal diffusivity.
γ	dimensionless constant in the Newtonian heating case.
δ	dimensionless parameter measuring temperature difference.
δ_c	critical value of δ.
θ	(dimensionless) temperature difference.
μ	fluid viscosity.
ρ	fluid density.
ρ_m	maximum fluid density.
ψ	(dimensionless) streamfunction.

heating whereby the surface heat flux is linearly related to the surface temperature. We identify a dimensionless parameter δ, which gives a measure of the temperature at its maximum density T_m in relation to the ambient temperature T_∞. The exact form that δ takes depends on the particular boundary conditions chosen and it is this parameter δ which determines the nature of the convective flow, with there being a critical value δ_c of δ which limits the range of steady solutions.

Convection flows in porous media are usually studied by assuming that the flow is driven either by a prescribed surface temperature or by a prescribed heat flux, see [1–3] for example. There has been previous work, for example by Ramanaiah and Malarvizhi [9], Ramanaiah and Kumaran [10] and Kumaran and Pop [11], that has considered the case of a prescribed heat transfer coefficient, a type of mixed boundary condition. A very limited amount of work has been done on problems with the type of mixed condition, known as 'Newtonian heating', initiated by Merkin [12] for a Newtonian fluid. Lesnic *et al.* [13] have investigated the steady free convection boundary-layer flow along a vertical surface embedded in a porous medium with Newtonian heating. Mixed convection boundary-layer flows in porous media with ambient temperatures at the density maximum [14,15] and close to the density maximum [16] have shown that the nature of these flows can be considerably different to the equivalent flow where a linear density–temperature relation applies. In [14–16] the outer flow and wall conditions were taken to allow the problem to be reduced to similarity form. In [16], where a density maximum could occur within the boundary layer, a dimensionless parameter equivalent to the δ used here arose and it was seen that, under certain external conditions, flows were possible only for a finite range of this parameter.

We start by deriving the equations for the steady states of our model, after which we consider the three separate cases of the wall conditions for the heat input describing numerical solutions to the resulting equations. As part of our discussion of the equations for the steady case we find critical values δ_c for the parameter δ with solutions possible only for $\delta \geq \delta_c$ and with the value of δ_c depending on the particular boundary conditions applied.

2. Equations

We consider the steady free convection flow near a stagnation point which is embedded in a fluid-saturated porous medium at ambient temperature T_∞ and density ρ_∞. We assume that $(\bar{x}, \bar{y})$ are cartesian co-ordinates along and normal to the surface, with corresponding velocity components $(\bar{u}, \bar{v})$. We take the density–temperature relation, proposed by [8], in the form

$$\rho(T) = \rho_0 + c_1 T - c_2 T^2 \tag{1}$$

where ρ and T are respectively the density and temperature of the convecting fluid and ρ_0, c_1 and c_2 are positive constants. From (1) the density has a local maximum ρ_m at temperature T_m, where

$$T_m = \frac{c_1}{2c_2}, \rho_m = \rho_0 + \frac{c_1^2}{4c_2}. \tag{2}$$

Cayley and McBride [8] tested the state Eq. (1) for pure water with temperatures ranging from 0 °C to 100 °C taking the coefficient values $\rho_0 = 999.845079$, $c_1 = 0.06378$ and $c_2 = 0.0080125$, with ρ in kg m^{-3} and T in 0 °C. They found that this expression gave $T_m \simeq 3.980$ °C, $\rho_m \simeq 999.97$kg m^{-3}, showing an increase from the density ρ_0 at T=0 °C. Note that at T=100 °C, the density is reduced to 926.1 kg m^{-3}.

The governing equations are, after using the boundary-layer approximation, Darcy's law for the flow and the density–temperature relation (1), see [1–3] for example,

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{3}$$

$$\bar{u} = \frac{gK}{\mu}\Big(c_2(T - T_\infty)^2 + (2c_2 T_\infty - c_1)(T - T_\infty)\Big)S(\bar{x}) \tag{4}$$

$$\bar{u}\frac{\partial T}{\partial \bar{x}} + \bar{v}\frac{\partial T}{\partial \bar{y}} = \alpha \frac{\partial^2 T}{\partial \bar{y}^2} \tag{5}$$

where $S(\bar{x}) = \frac{\bar{x}}{l}$ gives the shape of the boundary near the stagnation point and where K is the permeability of the porous medium, μ the viscosity of the convecting fluid, g the acceleration due to gravity and α the effective thermal diffusivity.

We consider three forms for the boundary conditions on $\bar{y} = 0$, namely

$$\text{(a) Prescribed wall temperature}: T = T_w$$

$$\text{(b) Prescribed wall heat flux}: \frac{\partial T}{\partial \bar{y}} = -q_w$$

$$\text{(c) Newtonian heating}: \frac{\partial T}{\partial \bar{y}} = -h_s T \tag{6}$$

where T_w, q_w and h_s are constants. As well as (6) we also have that

$$\bar{v} = 0 \quad \text{on} \quad \bar{y} = 0, \quad \bar{u} \to 0, \quad T \to T_\infty \quad \text{as} \quad \bar{y} \to \infty \tag{7}$$

We make Eqs. (3)–(6) dimensionless by writing