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# A non-singular boundary element method for modelling bubble dynamics in viscoelastic fluids

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ABSTRACT

When a cavity forms near a solid boundary a liquid jet can form directed towards the boundary, causing the generation of high pressures at the wall (potentially causing damage) and the formation of a toroidal bubble. In this paper several recent developments in the boundary element modelling of the dynamics of cavitation bubbles in viscoelastic fluids are presented. The standard formulation of the boundary element method (BEM) is in terms of a boundary integral equation with a singular kernel. A reformulation of the BEM in terms of a non-singular kernel is shown to provide enhanced stability. In situations when a liquid jet forms and impacts the far side of the bubble there is a transition to a toroidal form. This topological singularity in bubble geometry is modelled by placing a vortex ring inside the bubble to account for the circulation in the fluid and the discontinuity in potential following jet impact. The bubble dynamics are dependent on the initial stand-off distance from the boundary as well as the viscous and elastic properties of the fluid. It is shown that, while the viscosity of the fluid inhibits jet formation, the dynamics are particularly dependent on the relative strength of viscous, elastic and inertial forces. In particular, if the Deborah number is large enough elastic effects effectively negate fluid viscosity and behaviour similar to the inviscid case is recovered in terms of liquid jet formation.



## 1. Introduction

Despite their small size, cavitation bubbles can exhibit extreme physics with immense increases in pressure and temperature occurring during collapse. Their tendency to focus and concentrate energy, forces and stresses as well as emitting shockwaves means that they have the potential to cause damage to nearby surfaces and structures. This destructive behaviour has been utilised to advantage in a number of biomedical applications such as extracorporeal shock wave lithotripsy (ESWL) [35], ultrasound contrast imaging [9] and sonoporation [29]. An understanding of the behaviour of cavitation bubbles is essential to improve the effectiveness of each of these distinct procedures and to ensure that damage is restricted to the targeted areas.

The dynamics of an initially spherical bubble in an infinite extent of fluid was originally studied by Lord Rayleigh [37], motivated by the damage caused to ship's propellers from collapsing cavitation bubbles. The dynamics is described by the Rayleigh Plesset equation, the solution of which provides the evolution of the bubble radius.

Early theoretical work modelling a bubble near a rigid wall [36,46] was based on perturbations of the spherically symmetric solution developed by Rayleigh [37]. Later, Chahine and Bovis [12] extended this perturbation analysis to include the effects of surface tension using matched asymptotic expansions in powers of a small parameter $\epsilon$ defined by

$$\epsilon = \frac{R_m}{h}, \tag{1}$$

where $R_m$ and $h$ are the maximum bubble radius and distance from the centre of the bubble to the wall, respectively. However, this analysis is only valid for small values of $\epsilon$ and therefore is not applicable for the cases of interest in this paper where the bubble is near the wall for which $\epsilon \approx 1$. An alternative theoretical study was undertaken by Naude [31] who solved the Laplace equation for the velocity potential using Legendre polynomials and extended the theory to allow for larger perturbations.

The development of high-speed cameras allowed accurate photographs of bubble shape to be captured, the most notable early experimental study was that of Benjamin and Ellis [2]. Their experiments involved a Perspex sheet with cavities grown from nuclei situated at various small distances from it. The main phenomena captured in their experiments were the formation of a liquid jet in the direction of the rigid wall and the subsequent transition to a toroidal form. Benjamin and Ellis also seem to have been

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the first to realise the importance of the Kelvin Impulse in cavitation bubble dynamics. The Kelvin Impulse is the apparent inertia of the cavitation bubble and can be used to determine the direction of the bubble centroid and liquid jet [4]. Lauterborn and Bolle [23] measured jet velocities up to 120 m/s for a bubble near a solid plate and observed a small counterjet away from the boundary due to the bubble being driven towards the wall during collapse.

Early developments in numerical methods for bubble dynamics included the marker and cell method [30] which enabled the later stages of collapse to be predicted, beyond what was possible using perturbation techniques. The first fully numerical paper for describing the complete collapse of a cavitation bubble near a rigid wall was by Plesset and Chapman [34]. They developed a finite difference method based on cylindrical coordinates with the velocity potential determined from boundary conditions at the surfaces and at infinity. Their model demonstrated a remarkable agreement with the experiments of Lauterborn and Bolle [23] and, in particular, predicted the formation of a liquid jet.

To model a non-spherical bubble, the boundary element method (BEM) is often used. The BEM requires significantly less computational time and memory compared to other numerical methods such as finite elements or spectral elements since only the boundary is discretised. An additional advantage of BEM is that it is able to model the bubble surface as a true discontinuity obviating the need to employ sophisticated interface tracking techniques. The BEM was originally used to model a cavitation bubble by Guerri et al. [14]. It was further developed by Blake et al. [5,6] who considered the dynamics of an axisymmetric, vapour-filled bubble near a rigid wall and free surface. Since these early works a plethora of extensions of BEM have followed which have included the effects of buoyancy [43], elasticity [19] and viscoelasticity [27,42]. In terms of bubble topology extensions have included treatment of curved surfaces [41] and toroidal bubbles [3,44,47].

Improvements have also been made to BEM in terms of the accuracy of the discretisation of the bubble surface through the use of high-order (cubic and quintic) splines [25,42]. The improved accuracy of these spline discretisations mean that far fewer nodes are required to discretise the bubble surface, leading to improved computational performance. The standard BEM formulation in terms of a boundary integral equation contains kernels that are singular due to the divergence of the Green's function and its derivative around the source point. Additionally, when two nodes on the surface are close, near-singular behaviour leads to ill-conditioned linear systems. In this paper, a non-singular BEM formulation based on ideas of Klaseboer et al. [39] is developed for predicting bubble dynamics in the vicinity of a rigid wall. The non-singular formulation removes these singularities at the outset leading to a formulation of the BEM that is much more numerically stable.

The non-singular BEM formulation is found to dramatically reduce numerical errors produced by nodes becoming too close together. Consequently, the smoothing schemes typically used in the standard BEM formulation are no longer required to produce smooth bubble profiles. The use of quintic splines is shown to be more efficient than cubic splines in terms of the number of nodes required to attain a prescribed accuracy. The dynamics of a bubble in an Oldroyd-B fluid are found to be determined by a competition between viscous, elastic and inertial forces. Typically, viscous effects tend to reduce velocities and to inhibit jet formation although this can be negated by the elasticity of the fluid. For certain values of Reynolds and Deborah numbers a strong liquid jet occurs in the direction of the boundary, similar to the inviscid case. In contrast, however, the bubble centre is much thinner, resulting in the bubble rebounding away from the wall and negative pressures being generated.

## 2. Mathematical model

Consider a bubble initially spherical in shape and whose centroid is a distance $h$, known as the initial stand-off distance, from a rigid boundary of infinite extent. It is assumed that the bubble remains axisymmetric for all time, effectively reducing the dimension of the problem. Inherent in this assumption is that the bubble is stable to distortions from symmetry. Although this is not always the case, it is generally found to be true for small cavitation bubbles [7]. Additionally, the axisymmetric case can be seen as providing the instance of maximum jet speeds and pressures and thus is an indicator of maximum potential damage to nearby surfaces. It is also assumed that the fluid is incompressible and irrotational.

Since we are concerned with high speed bubble growth/collapse phenomena, it is reasonable to assume that the flow is inertia dominated in the bulk with viscous and viscoelastic effects being negligible. However, there are always thin boundary layers near the bubble where these effects can be appreciable due to the need to satisfy the physical stress boundary conditions. The thickness of the boundary layer depends on the competing influences of viscosity and elasticity and is approximately $1/\sqrt{(ReDe)}$. This justifies neglecting viscous diffusion when elastic effects are dominant and demonstrates that even for moderate $Re$ there is a return to inviscid behaviour in this case. Hence, the assumption that the entirety of the flow is irrotational with viscous and viscoelastic effects appearing through the normal stress balance at the bubble/free surface provides a consistent description of the physical problem. Despite not offering a solution to the full equations of motion, the irrotational assumption, at the very least, provides important and relevant insights into the dynamics of the problem.

In order to formulate a velocity potential, $\phi$, which satisfies the Laplace equation, it is necessary to assume incompressibility. The primary condition needed for this approximation to be valid is [1]

$$M^2 \ll 1, \tag{2}$$

where $M = \frac{U}{c}$ is the Mach number, $c$ is the speed of sound in the liquid and $U$ is the magnitude of variations of the fluid velocity with respect to both position and time. It is reasonable to assume incompressibility if $M^2 < 0.2$. Brujan [8] noted that in the late stages of collapse when a jet forms the bubble wall velocities can approach the speed of sound which means that the condition (2) is violated and liquid compressibility can no longer be ignored. These high velocities also give rise to very large pressures in the fluid.

Although methods based on potential theory predict initial bubble dynamics very well they can break down in the final stages of collapse when compressibility effects become important due to their inability to simulate shock waves, for example. Recent work has focused on developing methods capable of solving the Euler equations in order to handle shock waves and interfaces in a robust fashion. For example, So et al. [38] have developed an interface sharpening method for two-phase compressible flow simulations based on solving an anti-diffusion equation for the volume-fraction field that counteracts the numerical diffusion resulting from the underlying VOF discretization scheme. Johnsen and Colonius [16] have developed a high-order accurate shock- and interface-capturing scheme using a weighted essentially non-oscillatory (WENO) scheme to simulate the collapse of a gas bubble in water. However, an accurate treatment of compressible effects has yet to be incorporated into viscoelastic cavitation modelling.

Note that possible limitations of the incompressible model presented in this paper are that neither the shock wave generated at jet impact nor the wave reflected back into the bubble is modelled. Note that in real-life situations, a shock wave would be generated if the bubble initially expands at a speed larger than the speed