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Planar squeeze flow of a bingham fluid

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ABSTRACT

In this paper we study the planar squeeze flow of a Bingham plastic in the lubrication approximation. We assume that the domain occupied by the fluid is closed at one end and open at the other (planar geometry). We consider two cases: *(i)* planar walls approaching each other in a prescribed way; *(ii)* parallel walls whose shape depends on both time and longitudinal coordinate. The dynamics of the unyielded region is determined exploiting the integral formulation of the linear momentum balance. We prove that in proximity of the closed end the material is always yielded, so that the rigid part is always detached from it. When dealing with case *(ii)*, we show that the dynamics of the rigid domain is governed by a very complex integral equation, whose qualitative analysis is beyond the aims of this paper. Conversely, in case *(i)* we obtain an almost explicit solution.

1. Introduction

A Bingham (or viscoplastic) fluid is a material that behaves as rigid body for low stress values, or as a viscous fluid (whose viscosity may depend on the local strain rate) when the stress state exceeds a critical threshold (we refer the readers to the original papers by Bingham [3,4] or to [5]). As a consequence, unyielded regions that may stick to rigid walls or may be transported by the flow can develop within the material. In extreme cases, such regions may not exist at all or occupy the whole domain.

Modelling of Bingham materials has become increasingly important, especially because many materials encountered in industrial applications (e.g. foams, pastes, slurries, oils, ceramics, etc.) exhibit viscoplastic behaviour. One of the most cited application of the Bingham model is toothpaste, which visually exhibits the fundamental character of viscoplasticity: it flows (i.e. deforms indefinitely) only if submitted to a stress above some critical value, otherwise it behaves as a solid body.

Despite the apparent simplicity of the constitutive models (especially when formulated within the implicit constitutive theory [17–22]) the flow characteristics of these materials are difficult to predict, since they involve unknown boundaries separating the yielded and the unyielded regions. This is noticeably evident when considering specific settings such as squeeze flow or channel flow with non uniform walls. In particular the squeeze problem has been the subject of a series of papers of both experimental and theoretical nature. Here we mention [8,14,16,20,26,30] and the excellent paper [27], together with recent review by Coussot [7] and the numerous experimental papers therein cited.

When dealing with particular geometries that allow for major simplifications, such as the lubrication approximation [28], the Bingham model may lead to paradoxes or contradictions that invalidates the assumption of a perfectly rigid unyielded phase, [13]. This is the case, for instance, of the the well known "lubrication paradox", which essentially consists in the prediction of a plug speed that varies in the principal flow direction, meaning that a truly unyielded region cannot exist (see, for instance, [9,15,23] and [10]). This inconsistency has led some workers to consider strategies for overcoming the paradox. Balmforth and Craster [1] and subsequently Frigaard and Ryan [9] developed an asymptotic procedure that resolves the lubrication paradox and builds a consistent solution for thin layer problems. In practice they resolve the paradox by considering higher order terms of the lubrication expansion and by showing that actually the plugs are slightly above the yield stress. They call these regions pseudo plugs and they prove that true rigid plugs are embedded within them.

In the case of a squeeze flow the problem is still not exhaustively studied, see [2]. For this peculiar problem the lubrication paradox was first pointed out by Lipscomb and Denn in [13], who simply proved that a central unyielded rigid core cannot exist because of symmetry reasons. A major analysis of the squeeze flow paradox between parallel discs was performed in [29], where the Bingham model was viewed as a limiting case of a bi-viscous fluid and where it was proved that the limiting process tending to the Bingham model and the lubrication approximation lead to a contradiction. In a recent paper by Muravleva [16] the planar squeeze flow of a Bingham fluid is studied exploiting the asymptotic technique introduced in [1]. This

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technique, which has been successfully exploited by Frigaard et al. [9] for the flow in a channel with slowly varying width, allows to determine intact true plug regions, overcoming thus the lubrication paradox.

In this paper we study the same problem presented in Muravleva [16], but we use a different approach developed in [10]. In this approach, which traces its roots back to [24] and [21], the whole unyielded region is treated as an evolving non material volume, whose motion is determined only by the stress applied by the fluid part. In practice the balance of linear momentum of the unyielded region is written using the integral form of the momentum balance, where only the external stress (i.e. the force exerted by the fluid) acting on the boundary is required.

The advantage of our procedure lies in the fact that no assumption has to be made on the order of magnitude of the stress components when applying the lubrication scaling. In our opinion this is the correct way to proceed, since in the rigid domain the Cauchy stress is "indeterminate" and we cannot identify or verify a posteriori which term can be safely neglected when applying the scaling. The main result we get is that we are able to determine a "true" unyielded plug and a "true" yielded surface directly at the leading order with a plug speed that does not vary in the principal flow direction (no pseudo-plug or fake yield surfaces). Moreover, differently from the vast majority of studies on squeeze flow of Bingham plastics, we do not suppose that the velocity of the plates is constant and that the gap width in which the fluid is confined does not vary with time.

We study the squeeze flow between parallel plates that are approaching each other in a prescribed way, i.e. planar squeeze flow. We begin by considering a planar geometry in which one end is closed, while on the other a known uniform pressure is applied[1]. Then, in Section 4, we consider the more general case of time-dependent non-flat walls. We develop the model assuming that the ratio ε between the maximum channel width and the channel length is very small, i.e. the lubrication regime. Accordingly the flow equations are drastically simplified and explicit solutions can be found.

We prove that the unyielded part is always detached from the closed end of the channel and confined between the squeezing surfaces. Actually the yield condition is met also at the channel closed end and in a portion of the mid-plane, but both regions have at least $O(\varepsilon)$ measure, so that the microscopic dynamics occurring there cannot be observed at the leading order approximation. Our analysis is indeed confined to the leading order and models the flow on a length scale where $O(\varepsilon)$ variations are not observable. A higher order analysis may lead to the detection of unyielded parts even in proximity of the above mentioned regions, as proved in [12].

2. Derivation of the model

Let us consider the flow of an incompressible Bingham fluid in a channel of length[2] L^* and amplitude $2h^*(t^*)$, as depicted in Fig. 1. Because of symmetry, we confine our analysis to the upper part of the layer, namely $[0, h^*(t^*)]$. The velocity field is $\mathbf{v}^* = u^*(x^*, y^*, t^*)\mathbf{i} + v^*(x^*, y^*, t^*)\mathbf{j}$, where x^*, y^* are the longitudinal and transversal coordinate respectively.

The Cauchy stress is $\mathbb{T}^* = -P^*\mathbb{I} + \mathbb{S}^*$, where $P^* = 1/3\mathrm{tr}\mathbb{T}^*$ and $\mathbb{S}$ is the so-called deviatoric part. The Bingham constitutive equation can be written in the implicit form [17–22]

$$\mathbb{D}^* = \left(\frac{II_{\mathbb{D}^*}}{2\eta^* II_{\mathbb{D}^*} + \tau_o^*}\right)\mathbb{S}^*, \tag{1}$$

which automatically gives the mechanical incompressibility condition $\mathrm{tr}\mathbb{D}^* = 0$. In particular η^* is the viscosity, τ_o^* is the yield

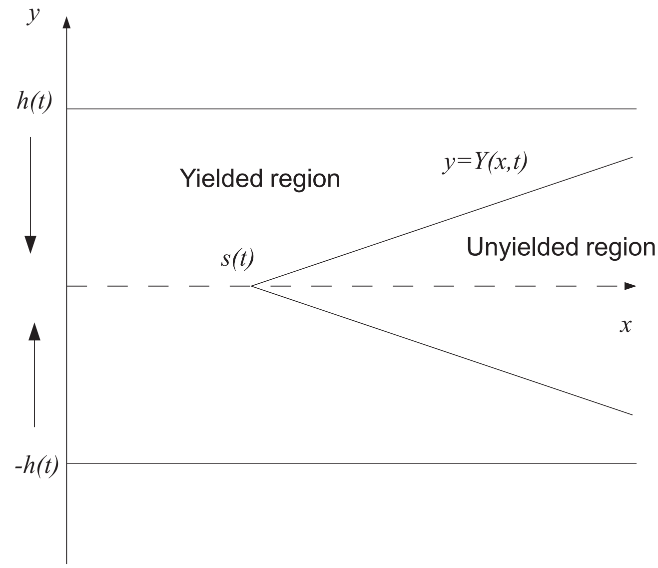

Fig. 1. A schematic representation of the squeezing channel.

stress and

$$\mathbb{D}^* = \frac{1}{2}\left(\nabla\mathbf{v}^* + \nabla\mathbf{v}^{*^{\mathrm{T}}}\right), \quad II_{\mathbb{S}^*} = \sqrt{\frac{1}{2}\,\mathrm{tr}\,\mathbb{S}^{*2}}, \quad II_{\mathbb{D}^*} = \sqrt{\frac{1}{2}\,\mathrm{tr}\,\mathbb{D}^{*2}}.$$

Eq. (1) allows to express $\mathbb{S}^*$ as a function of $\mathbb{D}^*$ only when $II_{\mathbb{S}^*} \geq \tau_o^*$, while $\mathbb{D}^* = 0$, $\Leftrightarrow$ $II_{\mathbb{S}^*} \leq \tau_o^*$, the stress being constitutively indeterminate.

We assume that the region where $II_{\mathbb{S}^*} \geq \tau_o^*$ (yielded) and the region where $II_{\mathbb{S}^*} \leq \tau_o^*$ (unyielded) are separated by a sharp interface $y^* = \pm Y^*(x^*, t^*)$ called the "yield surface". We also define the inner plug

$$\Omega^*_{p^*} = \left\{(x^*, y^*): \; x^* \in [0, L^*], \quad y^* \in [-Y^*, Y^*]\right\}.$$

Of course, it may occur that $Y^*(x^*, t^*) = 0$ for some $x^* \in (0, L^*)$ and/or for some t^*, so that $\Omega^*_{p^*}$ becomes a segment of zero measure. The rigid plug $\Omega^*_{p^*}$ moves uniformly and its velocity is

$$\begin{cases} u^* = u_p^*(t^*), \\ v^* = 0, \quad \text{(by symmetry)}. \end{cases} \tag{2}$$

Considering a quasi-steady dynamics and neglecting body forces, the governing equations in the viscous region are the mechanical incompressibility condition

$$\mathrm{tr}\mathbb{D}^* = 0,$$

and

$$-\frac{\partial P^*}{\partial x^*} + \frac{\partial S^*_{11}}{\partial x^*} + \frac{\partial S^*_{12}}{\partial y^*} = 0, \tag{3}$$

$$-\frac{\partial P^*}{\partial y^*} + \frac{\partial S^*_{12}}{\partial x^*} + \frac{\partial S^*_{22}}{\partial y^*} = 0, \tag{4}$$

where S^*_{ij} are the components of $\mathbb{S}^*$, given by (1), when $II_{\mathbb{S}^*} \geq \tau_o^*$.

The integral momentum balance for the whole domain $\Omega^*_{p^*}$, in the absence of body forces, is given by (see [11], [25] and [6])

$$\int_{\Omega^*_{p^*}} \frac{\partial}{\partial t^*}(\varrho^*\mathbf{v}^*)\mathrm{d}V^* + \int_{\partial\Omega^*_{p^*}} \varrho^*\mathbf{v}^*(\mathbf{v}^*\cdot\mathbf{n})\mathrm{d}S^* = \int_{\partial\Omega^*_{p^*}} (\mathbb{T}^*\mathbf{n})\mathrm{d}S^*, \tag{5}$$

where ϱ^* is the material density. Neglecting the inertial terms, we get following equation[3]

$$\int_0^{L^*} [-Y^*_{x^*}T^*_{11} + T^*_{12}]_{Y^{*+}}\,\mathrm{d}x^* + P^*_{Y_0}Y^*_0 - P^*_{Y_1}Y^*_1 = 0. \tag{6}$$

[1] We remark that assuming a "closed end" is equivalent to considering an open channel with symmetry condition.

[2] The starred variables indicate dimensional quantities.

[3] The expression $\left[-Y^*_x T^*_{11} + T^*_{12}\right]_{Y^{*+}}$ represents the force exerted by the viscous region on the lateral side of the inner rigid core.