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Equilibrium circulation and stress distribution in viscoelastic creeping flow

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ABSTRACT

An analytic, asymptotic approximation of the nonlinear steady-state equations for viscoelastic creeping flow, modeled by the Oldroyd-B equations with polymer stress diffusion, is derived. Near the extensional stagnation point the flow stretches and aligns polymers along the outgoing streamlines of the stagnation point resulting in a stress-island, or birefringent strand. The polymer stress diffusion coefficient is used, both as an asymptotic parameter and a regularization parameter. The structure of the singular part of the polymer stress tensor is a Gaussian aligned with the incoming streamline of the stagnation point; a smoothed δ-distribution whose width is proportional to the square-root of the diffusion coefficient. The amplitude of the stress island scales with the Wiessenberg number, and although singular in the limit of vanishing diffusion, it is integrable in the cross stream direction due to its vanishing width; this yields a convergent secondary flow. The leading order velocity response to this stress island is constructed and shown to be *independent* of the diffusion coefficient in the limit. The secondary circulation counteracts the forced flow and has a vorticity jump at the location of the stress islands, essentially expelling the background vorticity from the location of the birefringent strands. The analytic solutions are shown to be in excellent quantitative agreement with full numerical simulations, and therefore, the analytic solutions elucidate the salient mechanisms of the flow response to viscoelasticity and the mechanism for instability.

1. Introduction

Viscoelastic flows are found in many important engineering and biological systems. Despite the need to understand these flows in a variety of complex situations, analysis of the equations of motion describing viscoelastic fluids, even in the low-Reynolds number regime, is very incomplete. There are many different models depending on the rheology of the fluid, but little is known even for the simplest closed continuum models. One popular model, the Oldroyd-B model, can be derived from microscopic principles and represents "Boger" fluids, dilute solutions of polymers immersed in a Newtonian solvent which exhibit normal stress differences but not shear thinning. This model is used frequently in simulations of viscoelastic fluids even though there is no mathematical well-posedness theory for this system, i.e. it is not known if sufficiently smooth solutions to this system exist for all time, bringing in to question the reliability of any numerical simulation.

Flows at internal stagnation points (such as the four-roll mill flow or the cross-slot or cross channel flow) pose a particular difficulty for both theoretical investigation and numerical simulations of viscoelastic fluids, as polymers are aligned and stretched, and can create fine features in the flow that are difficult to resolve numerically. However it is precisely at these points in the flow that interesting dynamics arise. Instabilities have been found in experiments at internal stagnation points [1–5], and related numerical instabilities are found in similar geometries [6–11]. It is unclear what is driving these instabilities, but it is reasonable to conjecture that they are related to the large polymer stresses and stress gradients which accumulate along the incoming and outgoing streamlines of these internal stagnation points.

The elastic contribution to the total stress can be incorporated into the equations of motion by assuming that the total stress on the fluid, $\sigma = \boldsymbol{\tau}_s + \boldsymbol{\tau}_p$, comes from a solvent contribution $\boldsymbol{\tau}_s$ as well as a polymer contribution $\boldsymbol{\tau}_p$. In the case of a Newtonian solvent, the total stress is given by

$$\boldsymbol{\sigma} = -p\mathbf{I} + \eta_s \dot{\boldsymbol{\gamma}} + \boldsymbol{\tau}_p,$$

where η_s is the Newtonian solvent viscosity, and $\dot{\boldsymbol{\gamma}} = [\nabla \mathbf{u} + \nabla \mathbf{u}^T]$ is the rate-of-strain tensor. Assuming conservation of mass and incompressibility the fluid velocity $\mathbf{u}$ satisfies

$$\rho \frac{D\mathbf{u}}{Dt} = \nabla \cdot \sigma + \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0,$$

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for density ρ, and body force $\mathbf{f}$, or in the inertialess regime,

$$\nabla \cdot \sigma + \mathbf{f} = 0, \quad \nabla \cdot \mathbf{u} = 0. \tag{1}$$

In the Oldroyd-B model, the symmetric polymer stress tensor, $\boldsymbol{\tau}_p$, is advected via the upper-convected derivative and relaxes with a characteristic relaxation time λ:

$$\boldsymbol{\tau}_p + \lambda \overset{\nabla}{\boldsymbol{\tau}}_p = \eta_p \dot{\boldsymbol{\gamma}}. \tag{2}$$

Here η_p is the polymer viscosity, and the upper-convected derivative is defined by

$$\overset{\nabla}{\boldsymbol{A}} \equiv \frac{\partial \boldsymbol{A}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{A} - (\nabla \mathbf{u} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot (\nabla \mathbf{u})^T), \quad \text{where}$$

$$(\nabla \mathbf{u})_{ij} = \frac{\partial u^i}{\partial x_j}.$$

While Boger fluids are used in many experiments of viscoelastic phenomena, it is not immediately clear that the Oldroyd-B model is a good choice for modeling more general complex fluids. We choose to work with this model due to the generic nature of the upper-convected derivative. This represents a tensorial material derivative and hence will be found in continuum models which advect a macroscopic elastic stress tensor. Some other variants to the Oldroyd-B model include the Giesekus [12], Phan-Thien-Tanner (PTT) [13], and FENE-P models [14]. These models arise from different microscale models of the polymers. All of them introduce a nonlinear relaxation of stress which results in shear-thinning behavior. All of the above-mentioned macroscopic models contain the upper-convected derivative, the dominant source of nonlinearity in the equations, which leads to many of the difficulties and interesting phenomena associated with the Oldroyd-B model [8,9,15–17]. Oldroyd-B is the "simplest" of these models making it a good model for our theoretical work.

A simple modification to the Oldroyd-B model, which will yield smooth and bounded stresses [18,19], is to add polymer stress diffusion. The addition of stress diffusion can be derived from the kinetic theory of dumbbells [15,20], but the stress diffusion coefficient is proportional to the square of the ratio of the bead diameter (or polymer radius of gyration) to the flow length-scale, which even in the context of micro-fluidics is minute (on the size of 10^{-6} at most) [21]. To be useful as a regularization in numerical simulations, artificially large polymer stress diffusion is typically needed [18,22]. However it is useful to note that there is an analytical result [19] which proves that *any* amount of polymer stress diffusion will maintain a smooth and bounded polymer stress. In this manuscript we use polymer stress diffusion to derive an asymptotic expansion, in orders of the square-root of the stress diffusion coefficient, for solutions to the Oldroyd-B model (at zero Reynolds number) in a simple extensional flow geometry. This solution provides information about the effect of the large stress islands, or birefringent strands, on the resultant flow field. In particular, we are able to take the limit as the diffusion goes to zero and recover information about the effect of these stress islands on the flow. Therefore we can determine the first order effect of the stress island on the velocity in the Stokes-Oldroyd-B system.

An important structure of the momentum equation, which we use to guide us, is that at zero-Reynolds number the velocity is one degree smoother than the stress. This implies that at extensional points in the flow, where the stress accumulates, the exact value of the stress is not needed to determine the effect on the velocity. Only the integral of the stress affects the velocity field. The stress island can be approximated by a smoothed Dirac δ-distribution. Furthermore, when stress diffusion is included in the model, a Gaussian becomes an exact solution of the asymptotic approximation for the stress tensor.

The Gaussian has a well-defined integral even in the limit of zero diffusion which enables us to close the asymptotic expansion and give a well defined solution for the velocity. The result of the transversely narrow and sharply peaked stress distribution is a dip in the velocity whose magnitude is independent of the stress diffusion. Such a dip in the velocity field has been observed experimentally [23,24] and provides a possible mechanism for the instabilities seen in numerical simulations [6–11]. Simply stated, the instability mechanism is due to the fact that at extensional points in the flow the vorticity is low. In the low vorticity region, the stress can grow and where the stress is large the vorticity is expelled, leaving a larger area for the stress to begin to oscillate and become unstable. Boundary layer approximations near extensional stagnation points that depend on the polymer extension length and relaxation time were presented in [25,26].

In what follows we will describe the model and assumptions and derive an asymptotic expansion for the stress and velocity to first order in the stress diffusion coefficient. We conclude by showing that the solutions to our model agree extremely well with numerical simulations. The model captures both the leading order velocity response, as well as the amplitude of the stress in the birefringent strands.

1.1. Model

To perform the analysis it is simpler to write Eqs. (1)–(2) in terms of a conformation tensor, $\mathbf{S}$, defined by

$$\mathbf{S} = Wi\xi^{-1}\boldsymbol{\tau}_p + \mathbb{I}. \tag{3}$$

The addition of a polymer stress diffusion term, $\nu\Delta\mathbf{S}$ is added to the stress advection equation. This is necessary to our analysis, and we perform the asymptotic expansion in orders of the stress diffusion coefficient ν. In non-dimensional form we write the Stokes-Oldroyd-B equations with polymer diffusion as

$$\Delta\mathbf{u} - \nabla p + \xi Wi^{-1}\nabla \cdot \mathbf{S} + \mathbf{f} = 0, \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0, \tag{4}$$

$$Wi \overset{\nabla}{\mathbf{S}} + (\mathbf{S} - \mathbb{I}) = \nu\Delta\mathbf{S}. \tag{5}$$

The Weissenberg number, $Wi = \lambda/\tau_f$, is the ratio of the elastic relaxation time to the characteristic flow time-scale, set by $\mathbf{f}$ which we set to unity, and $\xi = \eta_p/\eta_s$ is the ratio of polymer to solvent viscosity.

1.2. Outline of solution strategy

The objective of this work is to find an analytic, asymptotic approximation of Eqs. (1)–(2) at steady state. Our analytical strategy has a few key steps which exploit both the structure of the upper convective derivative and the linearity of the stress feedback on the Stokes equations. Our steps will proceed as follows.

1. We rescale the velocity field by Wi, yielding a factor of Wi which multiplies the pressure and the force, $\mathbf{f}$. After the rescaling, Wi does not appear in the advection/diffusion equation for the stress.
2. In the rescaled variables, we choose a simple background flow, $\mathbf{u}$, to drive the dynamics of the upper convective derivative, without specifying the force, $\mathbf{f}$, which creates this flow. *Crucially* the flow we choose has the property $\partial_y u = 0$ everywhere. Physically speaking, this is a flow whose vorticity is zero near the maximum of the stress island. In constructing the solution, we will see that the stress feedback on the flow also produces a velocity field whose vorticity vanishes at the maximum of the stress island. Additionally, the feedback flow tends to expel vorticity from the vicinity of the maximum of the stress.