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Purely pulsating flow of a viscoelastic fluid in a pipe revisited: The limit of large Womersley number

R. Fernandez-Feria *, J. Alaminos-Quesada

Universidad de Málaga, Andalucía Tech, E. T. S. Ingeniería Industrial, Dr Ortiz Ramos s/n, 29071 Málaga, Spain

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ABSTRACT

The flow of a Maxwell fluid in a pipe generated by a pulsating pressure gradient is considered. The problem is governed by two non-dimensional parameters, the Womersley number α and a Deborah parameter γ. We obtain a simple approximate solution in the limit of large α (pipe diameter much larger than the viscous penetration depth) by solving the resulting boundary-layer problem through a two scales perturbation technique, which also helps to understand qualitatively the structure of the flow as γ increases. This limit is of interest, for instance, for the blood flow in arteries. The main qualitative differences between the Newtonian fluid flow ($\gamma = 0$) and the highly elastic fluid with $\gamma = O(1)$ or larger are easily discussed from the found solution in terms of just two algebraic functions of γ.

1. Introduction

The pulsating flow in pipes has been the subject of many studies for a long time since it is of interest in many biological and technological systems and processes. In particular, analytical solutions for the Newtonian fluid flow in a rigid infinite pipe due to a periodic pressure gradient were considered by Womersley [1] in relation to the blood flow in arteries, although similar solutions were found earlier [2,3]. Since the blood is known to behave as a non-Newtonian fluid [4], analytical solutions for several non-Newtonian fluid models have also been obtained later for this idealized pipe flow, as well as for some other more general pulsating pipe flows, not only in relation to blood flow, but to some other physiological and industrial processes, many of them in relation to the flow enhancement effect [5–14]. Some of these works, in addition to other ones for the viscoelastic fluid flow generated by an oscillating plate [15,16], are cited below in connection to the solutions described in the present work. Here we consider the flow in a pipe due to a purely periodic pressure gradient of a fluid modelled by a simple Maxwell model in order to analyse the interesting pulsating behavior of a viscoelastic fluid in the limit of large Womersley numbers, of interest for the blood flow in arteries. To that end we obtain simple analytical approximations of the velocity field and the flow rate in that limit (Section 3) which allows for a better understanding of the flow.

2. Formulation of the problem

We consider the unidirectional and incompressible flow in a circular pipe of infinite length and radius R, governed by the momentum equation in the flow direction x,

$$\rho \frac{\partial u}{\partial t} = p_l + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rx}), \tag{1}$$

and boundary conditions

$$u(R,t) = 0, \quad u(0,t) \neq \infty, \tag{2}$$

where ρ is the fluid density, $u(r,t)$ the longitudinal velocity, r the distance from the axis of the pipe and t the time, when the pressure gradient p_l is a periodic function of time, e.g.,

$$p_l(t) = A\cos(\Omega t), \tag{3}$$

for given frequency Ω and amplitude A, and the fluid has a viscoelastic behavior given by a simple Maxwell model for the shear stress component τ_{rx} [17],

$$\tau_{rx} = \frac{\eta}{\lambda}\int_{-\infty}^{t} e^{-\frac{t-t'}{\lambda}} \frac{\partial u(r,t')}{\partial r}\, dt', \tag{4}$$

where η is the zero-shear viscosity and λ the relaxation time.

Defining the non-dimensional variables

$$\xi = \frac{r}{R}, \quad \tau = \Omega t, \quad v = \frac{\rho\Omega}{A} u, \tag{5}$$

and writing (3) as $p_l = \Re\left[Ae^{i\Omega t}\right]$, Eqs. (1)–(4) can be written as

* Corresponding author. Tel.: +34 951952380.
E-mail address: ramon.fernandez@uma.es (R. Fernandez-Feria).

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$$\frac{\partial w}{\partial \tau} = e^{i\tau} + \frac{1}{\alpha^2 \gamma} \frac{1}{\xi} \frac{\partial}{\partial \xi} \left(\xi \int_{-\infty}^{\tau} e^{-\frac{\tau - t'}{\gamma}} \frac{\partial w(r, t')}{\partial \xi} d\tau' \right), \tag{6}$$

$$w(1, \tau) = 0, \quad w(0, \tau) \neq \infty, \tag{7}$$

with

$$v = \Re[w], \tag{8}$$

and where we have defined the non-dimensional parameters

$$\alpha \equiv \frac{R}{\sqrt{\eta/(\rho\Omega)}}, \quad \gamma \equiv \lambda\Omega, \tag{9}$$

which are usually called Womersley and Deborah numbers, respectively. We look for periodic solutions in the form

$$w(\xi, \tau) = f(\xi) e^{i\tau}. \tag{10}$$

Particularly, we look for simple analytic approximate solutions in the limit $\alpha \gg 1$ with $\gamma = O(1)$ or larger. Note that this limit of large Womersley number ($\alpha \gg 1$) corresponds to a 'wide' pipe, in the sense that its radius R is much larger than the viscous penetration depth $\sqrt{\eta/(\rho\Omega)}$. It must be emphasized that we only consider here the laminar, unidirectional flow in a cylinder of infinite length. The hydrodynamic stability of the solutions found is not considered, and it is known that the viscoelastic fluid flow in a pipe may be hydrodynamically unstable even for negligible inertial forces (e.g., [18]).

3. Solution in the limit $\alpha \gg 1$

3.1. Review of the exact solution for any α

Substituting (10) into (6) and (7) one gets

$$f'' + \frac{f'}{\xi} - i\alpha^2(1 + i\gamma) f = -\alpha^2(1 + i\gamma), \tag{11}$$

$$f(1) = 0, \quad f(0) \neq \infty, \tag{12}$$

where primes denote differentiation with respect to ξ. The analytical solution to this problem can be written as

$$f(\xi) = i\left(\frac{J_0(\alpha\sqrt{\gamma - i}\,\xi)}{J_0(\alpha\sqrt{\gamma - i})} - 1 \right), \tag{13}$$

where J_0 is the zero order Bessel function of the first kind [19]. Thus, the non-dimensional velocity field can be written as

$$v(\xi, \tau) = \Re\left[\frac{e^{i\tau}}{i} \left(1 - \frac{J_0(\alpha\sqrt{\gamma - i}\,\xi)}{J_0(\alpha\sqrt{\gamma - i})} \right) \right]. \tag{14}$$

This solution was found, in dimensional form, by Broer [5], and contains that for a Newtonian fluid when $\gamma = 0$, found earlier by some other authors in several forms [2,3,1]:

$$v(\xi, \tau) = \Re\left[\frac{e^{i\tau}}{i} \left(1 - \frac{J_0(\alpha\xi i^{3/2})}{J_0(\alpha i^{3/2})} \right) \right]. \tag{15}$$

Actually, Broer [5] also obtained the next corrections to this expression due to the viscoelastic behavior by expanding (14) for $\gamma \ll 1$.

3.2. Approximate non-dimensional velocity field for $\alpha \gg 1$ from the method of two scales

Here we are interested in the behavior of the flow for any γ when α is large, which presents some interesting features worth to be discussed using a simpler and more explicit expression than (14).

For large α, viscous effects are confined in a narrow region close to the pipe wall; i.e., the solution presents a boundary-layer structure near $\xi = 1$. Thus, the approximate solution in this limit can be obtained either by using the asymptotic limit of the Bessel functions in (14) for $\alpha \gg 1$, or by solving (11) approximately using a singular perturbation technique. Both methods are obviously equivalent. Since the perturbation technique applied to the original equation and boundary conditions is both analytically simpler and physically more informative about the structure of the solution, we adopt here this second approach.

Writing (11) as

$$\beta f'' + \frac{\beta}{\xi} f' - if = -1, \tag{16}$$

with

$$\beta = \frac{1 - i\gamma}{\alpha^2(1 + \gamma^2)}, \tag{17}$$

we look for solutions when $|\beta| \ll 1$. In this limit, the solution has a boundary layer near $\xi = 1$ of thickness $\beta^{1/2}$. To obtain the inner solution inside this boundary layer one uses the appropriate inner variable

$$\xi_i = \frac{\xi - 1}{\beta^{1/2}}, \tag{18}$$

and matches it to the external solution. It is easily found from (16) that this external solution is $f_e = -i$ in all the orders of the expansion in powers of β. Thus, the external solution for the velocity, valid to any order in β, is, according to (8) and (10),

$$v_e = \Re[-ie^{i\tau}] = \sin\tau. \tag{19}$$

Note that this solution, valid in the bulk of the pipe flow but not in the vicinity of the pipe wall $\xi = 1$, is not in phase with the pressure gradient oscillation $\cos\tau$, having a phase shift of $\pi/2$.

To obtain a uniform valid solution across all the pipe section one has to match asymptotically the inner solution in terms of ξ_i, valid near $\xi = 1$, to the external solution $f_e = -i$. However, it turns out that the inner asymptotic expansion contains *secular* terms and fails to converge appropriately, so that the matched asymptotic expansions technique fails. To avoid this, we use a method of two scales to solve the boundary layer problem [20]. We define

$$f(\xi) = f(\xi_e, \xi_i), \tag{20}$$

where $\xi_e = \xi$ and ξ_i is defined in (18). Using the chain rule to obtain $df/d\xi$ and $d^2f/d\xi^2$ and substituting into (16) one obtains

$$\beta\left(\frac{\partial^2 f}{\partial \xi_e^2} + \frac{2}{\beta^{1/2}} \frac{\partial^2 f}{\partial \xi_e \partial \xi_i} + \frac{1}{\beta} \frac{\partial^2 f}{\partial \xi_i^2} \right) + \frac{\beta}{\xi_e}\left(\frac{\partial f}{\partial \xi_e} + \frac{1}{\beta^{1/2}} \frac{\partial f}{\partial \xi_i} \right) - if = -1, \tag{21}$$

which has to be solved with the boundary conditions

$$f(1, 0) = 0, \quad f(0, -\beta^{-1/2}) \neq \infty. \tag{22}$$

Substituting the asymptotic expansion

$$f \sim f_0 + \beta^{1/2} f_1 + \ldots, \tag{23}$$

into (21), at the lowest order one has

$$\frac{\partial^2 f_0}{\partial \xi_i^2} - if_0 = -1, \tag{24}$$

whose general solution is

$$f_0 = C_{01}(\xi_e) e^{\frac{1+i}{\sqrt{2}}\xi_i} + C_{02}(\xi_e) e^{-\frac{1+i}{\sqrt{2}}\xi_i} - i, \tag{25}$$

where C_{01} and C_{02} are arbitrary functions of ξ_e to be obtained at the next order in the expansion.

At the next order $\beta^{1/2}$ one has