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A rational derivation of dynamic higher order equations for functionally graded micropolar plates

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ABSTRACT

The dynamics of functionally graded micropolar plates is considered. The derivation process is based on power series expansions in the thickness coordinate. Using the three-dimensional equations of motion for micropolar continuum, variationally consistent equations of motion and end boundary conditions are derived in a systematic fashion up to arbitrary order. Numerical results are presented for simply supported plates using different material distributions for both low and high order truncation orders. These results illustrate that the present approach renders benchmark solutions provided higher order truncations are used, and act as engineering plate equations using low order truncation.

1. Introduction

Functionally graded (FG) materials are composite materials made of two (or more) phases of material constituents, where the phase distribution varies continuously. The most used group of FG materials consists of ceramic and metal phases. Such materials were developed in the mid 1980s where the strength of the metal and the heat resistance of the ceramic made these materials well suited for high-temperature environments. FG materials also possess a number of further advantages compared to other inhomogeneous materials such as improved residual stress distribution, higher fracture toughness, and reduced stress intensity factors. Hence, FG materials are nowadays used in many different fields of engineering [1,2].

FG plates using classic elastic continuum theory have been studied extensively in recent decades. Among these, work on FG plates using three dimensional or higher order two dimensional theories are found in [3–9]. Several comprehensive surveys on various aspects of FG plate modeling have been reported [10–13].

In micropolar elasticity theory the classical continuum model is extended to properly deal with microscale effects which affects the mechanical response e.g. in granular or fibrous materials. In addition to the classical theory where three translational degrees of freedom are assigned to each material point, the micropolar theory adds three rotational degrees of freedom to each material point. This results in a so called couples stress tensor in addition to the regular stress tensor, where both these tensors are nonsymmetric in the general case. Among the different existing micropolar theories based on the Cosserat work [14], the version developed by Eringen [15] is perhaps the most popular. There are several work on micropolar plates adopting Eringen's theory [16–23], see also the review paper [24].

The present work on FG micropolar plates is an extension to the work on dynamic equations for homogeneous micropolar plates [23]. The method used renders a hierarchy of micropolar FG plate equations in a consistent and systematic fashion up to arbitrary order. The method is mainly based on the works on homogeneous plates that are isotropic [25], anisotropic [26] and piezoelectric [27], and has also been employed for functionally graded isotropic plates [3] adopting classic elastic continuum theory. The interest towards FG structures taking microscale effects into account has developed over the last few years. The majority of studies are based on the modified couple stress theory [28–32] and other alternative theories [33,34]. However, there are to our knowledge no work on FG plates using the more general micropolar theory, and thus this work may contribute to fill that gap.

The derivation process for developing plate equations for FG isotropic micropolar plates is here based on employing a systematic power series expansion approach using the three dimensional equations of motion for micropolar continuum. Displacement and micro-rotation fields, as well as material parameters, are expanded in a power series in the thickness coordinate of the plate. From these expanded fields, the stress and couple stress are obtained on power series form in terms of the expansion functions of the

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displacements and micro-rotations. Furthermore, by using the equations of motion for micropolar elasticity, recursion relations are constructed. These are used to express all expansion functions in terms of the lowest order expansion functions. Thus all fields can be expressed in these lowest order expansion functions without performing any truncations. Subsequently boundary conditions on the upper and lower surface of the plate are stated on power series form. These boundary conditions represent a set of scalar equations, written in terms of the lowest order expansion functions, and constitute the complete set of partial differential plate equations. Using variational calculus, the edge boundary conditions for each edge surface are obtained in an equally systematic manner. The resulting sets of plate equations may be truncated to any desired order. Numerical results are presented for simply supported plates where the material distribution is varying using a power law through the plate thickness. The results comprise eigenfrequencies and cross sectional fields using different truncation orders. The low order cases may be used as approximate engineering plate theories while the higher order theories act as benchmark theories converging to the exact 3D solution.

2. Theory of linear micropolar elasticity

Consider an isotropic micropolar continuum according to Eringen's theory [15]. The equations of balance of momentum and moment of momentum, written in cartesian coordinates, are expressed as

$$t_{kl,k} = \rho \ddot{u}_l, \tag{1}$$

$$m_{kl,k} + \epsilon_{lkm} t_{km} = \rho j_{lk} \ddot{\phi}_k, \tag{2}$$

in absence of body forces and body couples. Here t_{kl} is the stress tensor, m_{kl} is the couple stress tensor, u_l is the displacement vector, ϕ_k is the micro-rotation vector, ρ is the density, j_{lk} is the microinertia tensor and ϵ_{lkm} is the permutation symbol. Indices that follow a comma indicate partial differentiation. The surface tractions are defined in accordance to

$$t_l = t_{kl} n_k, \tag{3}$$

$$m_l = m_{kl} n_k, \tag{4}$$

where n_k is an outward pointing normal surface vector.

The micropolar strain tensors ε_{kl} and γ_{kl} are defined by

$$\varepsilon_{kl} = u_{l,k} + \epsilon_{lkm} \phi_m, \tag{5}$$

$$\gamma_{kl} = \phi_{k,l}. \tag{6}$$

These strain measures are related to the stress and couple stress tensors through the constitutive relations

$$t_{kl} = \lambda \varepsilon_{mm} \delta_{kl} + (\mu + \kappa) \varepsilon_{kl} + \mu \varepsilon_{lk}, \tag{7}$$

$$m_{kl} = \alpha \gamma_{mm} \delta_{kl} + \beta \gamma_{kl} + \gamma \gamma_{lk}, \tag{8}$$

where δ_{kl} is the Kronecker delta, λ and μ are Lamé parameters while α, β, γ and κ are micropolar elastic moduli. Consider from now on spin-isotropic materials where the microinertia reduces to a scalar quantity, $j_{kl} = j\delta_{kl}$.

3. Series expansion and recursion relations

A hierarchy of approximate equations for isotropic plates that are FG in the thickness direction is to be derived in a consistent manner based on governing equations for a micropolar continuum as described in Section 2. Consider a plate of thickness $2h$ using a cartesian coordinate system $\{x, y, z\}$, where the in plane x and y axes are along the middle plate plane at $z = 0$. The components of the displacement field and micro-rotation field are denoted $\{u_1, u_2, u_3\}$ and $\{\phi_1, \phi_2, \phi_3\}$ respectively. The derivation procedure of the plate equations is based on the assumption that each component of the displacement field and micro-rotation field can be expanded in a power series in the thickness coordinate z according to

$$u_l(x, y, z, t) = \sum_{n=0}^{\infty} z^n u_l^{(n)}(x, y, t), \tag{9}$$

$$\phi_l(x, y, z, t) = \sum_{n=0}^{\infty} z^n \phi_l^{(n)}(x, y, t), \tag{10}$$

for $l = 1, 2, 3$. As for the material parameters varying in the thickness direction, these are expanded in Taylor series [3] as

$$f(z) = \sum_{n=0}^{\infty} z^n f^{(n)}. \tag{11}$$

Here, f covers both traditional elastic parameters $\{\rho, \lambda, \mu\}$ and the micropolar parameters $\{\kappa, \alpha, \beta, \gamma, j\}$, see further discussions in Section 5.

By using the series expansions Eqs. (9)–(11) into the deformation relations Eqs. (5) and (6), the stress and couple stress expressions from the constitutive relations Eqs. (7) and (8) are written on power series form

$$t_{kl} = \sum_{n=0}^{\infty} z^n t_{kl}^{(n)}, \tag{12}$$

$$m_{kl} = \sum_{n=0}^{\infty} z^n m_{kl}^{(n)}. \tag{13}$$

Each power series term may thus be expressed in terms of displacements and rotations through

$$t_{kl}^{(n)} = \left[\lambda \bigstar L_j u_j\right]_n \delta_{kl} + \left[\mu \bigstar (L_l u_k + L_k u_l)\right]_n + \left[\kappa \bigstar (L_k u_l + \epsilon_{lkm} \phi_m)\right]_n, \tag{14}$$

$$m_{kl}^{(n)} = \left[\alpha \bigstar L_j \phi_j\right]_n \delta_{kl} + \left[\beta \bigstar L_l \phi_k\right]_n + \left[\gamma \bigstar L_k \phi_l\right]_n. \tag{15}$$

Here the product rule for power series are denoted

$$[a \bigstar b]_n = \sum_{i=0} n a^{(i)} b^{(n-i)}. \tag{16}$$

Moreover, operators L_k acting on the series expanded fields are introduced in order to simplify the expressions according to

$$L_k g_l^{(n)} = \begin{cases} \partial_x g_l^{(n)} & \text{if } k = 1, \\ \partial_y g_l^{(n)} & \text{if } k = 2, \\ (n+1) g_l^{(n+1)} & \text{if } k = 3, \end{cases} \tag{17}$$

where $g_l^{(n)}$ is any of the expansion fields in Eqs. (9)–(11). Hence L_k for $k = 3$ increases the index of $g_l^{(n)}$ and multiplies with the new index, in this case $n + 1$. Note the shorthand form ∂_x and ∂_y used to denote partial derivatives with respect to x and y.

Now that the material parameters, displacements, micro-rotations, stresses and couple stresses are all expressed on power series form according to Eqs. (9)–(13), these fields are to be used in the equations of motion, Eqs. (1) and (2). Collecting terms of equal power in z in the equations of motion results in recursion formulas for each displacement and micro-rotation field according to

$$\begin{aligned} &(n+1)[(\mu+\kappa) \bigstar L_3 u_l]_{n+1} + (n+1)[(\lambda+\mu) \bigstar L_3 u_3]_{n+1} \delta_{3l} \\ &\quad = [\rho \bigstar \ddot{u}_l]_n - \left[(\mu+\kappa) \bigstar (\partial_x^2 + \partial_y^2) u_l\right]_n - L_l\left[\lambda \bigstar (\partial_x u_1 + \partial_y u_2)\right]_n \\ &\quad - \left[\mu \bigstar L_l (\partial_x u_1 + \partial_y u_2)\right]_n - (n+1)[\mu \bigstar L_l u_3]_{n+1}(1 - \delta_{3l}) \\ &\quad - [\lambda \bigstar L_l L_3 u_3]_n (1 - \delta_{3l}) - \epsilon_{ijk} L_j [\kappa \bigstar \phi_k]_n \delta_{li}, \end{aligned} \tag{18}$$