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An analytical solution for viscoelastic dean flow in curved pipes with elliptical cross section

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ABSTRACT

In this paper, a new analytical solution for viscoelastic flow in curved elliptical pipes is presented for the first time. The perturbation method is used to derive the analytical solution, and the curvature ratio is considered as the perturbation parameter. The Oldroyd-B model is used as the constitutive equation, so the result of the present study could be useful for modeling the flow of dilute polymeric solutions inside curved elliptical pipes. The analytical solution is derived using an appropriate transformation that converts the elliptical shape of a cross section to the unit circle. The transformed governing equations are solved to the second-order terms using the perturbation method and the velocity field is obtained by implementing the inverse transformation on the results. Here, the effects of geometry, Weissenberg number and Reynolds number on the axial velocity, secondary flows and flow rate are studied in detail.

1. Introduction

Fluid flow in curved pipes is one of the most important flows in fluid mechanics, as it has practical applications in industry such as (1) fluid flow in pipelines, (2) internal cooling of turbine blades, (3) flow in biological systems and (4) ducting in internal combustion engines and (5) heat exchangers. The first analytical solution of Newtonian flow in a curved pipe with circular cross section was performed by Dean [1,2], who used the perturbation method and found the onset of a pair of counter-rotating vortexes due to the centrifugal force. Dean introduced a dimensionless number which specifies the behavior of flow within curved ducts. This dimensionless group represents the square root of the ratio between the product of the inertia and centrifugal forces to the viscous force, and measures the intensity of the secondary flow. Topakoglu [3] improved Dean's work by calculating the higher order terms of perturbation series for flow-field parameters and obtaining an analytical relationship for the flow rate of Newtonian fluid in a curved pipe. Similar investigations of Newtonian flow in circular curved pipes have been performed by other researches [4–12].

Most previous analytical research has focused on the flow in curved circular and annular pipes. Only limited research is available regarding other cross-section shapes due to the mathematical difficulties of these problems. Flows in curved noncircular ducts are of increasing importance in microfluidics, where lithographic methods typically produce channels with noncircular cross-section shapes. These channels are widely used in biological kits (e.g. DNA extraction, cancers cell and bacteria detection, blood sample preparation, and glucose monitoring kits), fuel cells, and compact heat exchangers for small scales. Limited research regarding Newtonian flow in curved pipes with an elliptical cross section includes Cuming [13] who found non-axisymmetric secondary flows using Dean's [1,2] approach. Topakoglu and Ebadian [14] also studied flow in curved pipes of elliptical cross section for two geometries in which major or minor axis of the ellipse is in the radius of curvature. They introduced an expression for the first term of expansion of the secondary flow as a function of the ellipticity ratio. In another research [15], they extended their previous work and introduced a rate-of-flow expression for any value of flatness ratio of the elliptical cross section. Takami and Sudou [16] investigated fluid flow in an elliptically curved pipe both analytically and experimentally. They compared their experimental results to analytical results on pressure drop. They also suggested a correlation for the friction coefficient. Tuttle [17] found the secondary flow pattern of a pipe with an elliptical cross section whose axis is straight but which is twisted about its axis. He also revised the results of previous studies regarding helical pipes. Other researchers have studied this flow by focusing on the orientation of the cross section and Dean's number [18–23].

Previous theoretical and numerical works regarding non-Newtonian Dean flow generally have been performed for circular-shaped of cross sections. Zhang et al. [24] used Galerkin's method to study Oldroyd-B fluid flow in circular curved pipes.

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They obtained the analytical solution of flow field at large Dean and Weissenberg numbers. Fan et al. [25] used finite volume method to find a solution for fully developed creeping and inertial flow of Oldroyd-B and UCM fluids in a curved circular pipe. They showed that the first normal stress difference amplifies the intensity of secondary flows and decreases the pressure drop value. Their results have good agreement with experimental results [26–28]. Besides numerical investigations, analytical solutions especially, via the perturbation method, are used to study the flow of non-Newtonian fluids in circular curved pipes. The first research regarding viscoelastic fluids was performed by Thomas and Walters [29]. They studied the fully developed flow of viscoelastic fluid in a curved pipe by the perturbation technique and showed that the intensity of secondary flows has a direct relationship with elastic properties of Oldroyd-B fluid. Bowen et al. [30] used Upper Convected Maxwell (UCM) and second-order models to obtain the flow rate of creeping flow in curved circular pipes. Robertson and Muller [31] used perturbation methods to study fully developed flow of Oldroyd-B fluids through curved pipes with circular and annular cross-sections. Their results showed that in pipes of circular cross-section, the velocity field for creeping flows of Oldroyd-B fluids is qualitatively similar to non-creeping flows of Newtonian fluids and a pair of counter-rotating vortices is generated in the flow field. They also showed that in curved annular pipes, because of inertial or elastic effects, two pairs of counter-rotating vortices exist in the flow field. Additionally, they investigated the effects of elasticity on the drag for non-zero Reynolds number. Similar studies have been devoted for Reiner–Rivlin fluid [32], Bingham fluid [33,34], second-order fluid [35,36] and Oldroyd-B fluid in rotating curved pipes [37] using Dean's approach.

Limited research has been focused on the viscoelastic flow in curved pipes with elliptical cross section. Thomas and Walters [38] studied the flow of an elastico-viscous liquid in a curved pipe under a pressure gradient with an elliptical cross section. They showed that counter-rotating vortices are generated in this flow. They also showed that the flux through the pipe is independent of the curvature of the pipe. They did not use suitable function for viscoelastic effects and could not see the effects of elasticity in the fluid correctly [39].

Their solution only obtained functional form of main flow velocity with a complicated coefficient which they could not solve and so they could not calculate flow rate. They could only calculate the stream line with first order of perturbation solution. Sarin [40], using the same perturbation-method studied the fully developed steady laminar flow of an idealized elstico-viscous liquid through a curved tube with an elliptical cross section. He found that and the cross-sectional area varies slowly with longitudinal distance. He reported functional forms of velocity components involve a large number of unknowns in terms of basic parameters in which numerical calculations should be carried out for calculating the values of parameters. This makes the general discussion of the behavior of the velocity profiles quite complicated. He did not report any graph regarding streamline and velocity profile and flow rate values He concluded that in a tube of increasing curvature, secondary flows generated with delay. He also showed that the point of maximum shear stress varies with the cross section.

As mentioned before, the most studies have been focused on Newtonian fluids and only a few works are available on the non-Newtonian fluid flow in channels with elliptical cross section. Unlike previous studies, a new analytical solution for fully developed Oldroyd-B fluid flow in a curved pipe with elliptical cross section is presented using perturbation method. Generally, the current study is the generalization of the work of Robertson and Muller [31] for viscoelastic flow in elliptical curved pipes. This method was not used previously, and we found our solution by applying a suitable mapping to change the domain of solution from elliptical form to unit circle. The mapped governing equations are solved, and the solution is obtained up to second order of perturbations series for the first time. Besides mentioned applications, this solution can be especially used for validation of CFD codes and sometimes for calibration of experimental setups. Fig. 1 shows the geometry of a curved pipe which is used in current research. Here, the toroidal coordinate system is used for studying the flow. According to the Figure, R is the pitch radius of curvature.

Due to the complicated solution, MAPLE software is used for deriving the flow solution. It is important to emphasize that it is not possible to find a relationship for flow rate based on the first-order terms of perturbation series. By calculating second-order terms, an analytical relation for the flow rate of the Oldroyd-B fluid in a curved pipe is obtained. Also, in the presence of higher-order terms, the axial velocity and secondary flows are calculated more accurately.

2. Mathematical modeling

2.1. Non-dimensional parameter

The non-dimensional parameters of the current study are:

$$\begin{aligned}
&s=\frac{\tilde{s}}{d_h/2},\quad r=\frac{\tilde{r}}{d_h/2},\quad x=\frac{\tilde{x}}{d_h/2},\quad y=\frac{\tilde{y}}{d_h/2},\quad D=\tilde{D}\frac{d_h/2}{W_{ref}}\\
&a=\frac{\tilde{a}}{d_h/2},\quad b=\frac{\tilde{b}}{d_h/2},\quad u=\frac{\tilde{u}}{W_{ref}},\quad v=\frac{\tilde{v}}{W_{ref}},\quad w=\frac{\tilde{w}}{W_{ref}},\\
&v_r=\frac{\tilde{v}_r}{W_{ref}},\quad v_\theta=\frac{\tilde{v}_\theta}{W_{ref}},\quad v_s=\frac{\tilde{v}_s}{W_{ref}},\quad \sigma_1=\frac{d_h/2}{\eta W_{ref}}\tilde{\sigma}_1,\quad \sigma_2=\frac{d_h/2}{\eta W_{ref}}\tilde{\sigma}_2,\\
&\mathrm{Re}=\frac{\rho W_{ref} d_h/2}{\eta},\quad We=\frac{\lambda W_{ref}}{d_h/2},\quad \kappa=\frac{d_h/2}{R},\quad Dn=Re\sqrt{\kappa}
\end{aligned} \tag{1}$$

where r and s are the components of polar-toroidal coordinate system, W_{ref} is the reference velocity, u is the velocity component in x direction, v is the velocity in y direction, w is the axial velocity in φ direction, v_r is the velocity component in r direction, v_θ is the velocity component in θ direction, v_s is the velocity component in s direction, $\tilde{\sigma}_1$ is the polymeric stress, $\tilde{\sigma}_2$ is the Newtonian solvent stress, λ is the relaxation time of fluid, η is the viscosity, D is rate of deformation tensor, Re is Reynolds number, κ is the curvature ratio, R is the curvature radius of the curved pipe, and We is Weissenberg number. Also d_h is the hydraulic diameter of pipe defined as follows:

$$d_h=\frac{4\tilde{A}}{\tilde{P}}=\frac{4(\pi\tilde{a}\tilde{b})}{2\pi\left(\frac{1}{2}(\tilde{a}^2+\tilde{b}^2)\right)^{1/2}}=\frac{2\tilde{a}\tilde{b}}{\left(\frac{1}{2}(\tilde{a}^2+\tilde{b}^2)\right)^{1/2}} \tag{2}$$

In Eq. (2), $\tilde{A}$ is the cross-sectional area and $\tilde{P}$ is the perimeter of the cross section. Also $\tilde{a}$ and $\tilde{b}$ are the major and minor axis of the elliptical cross section, respectively.

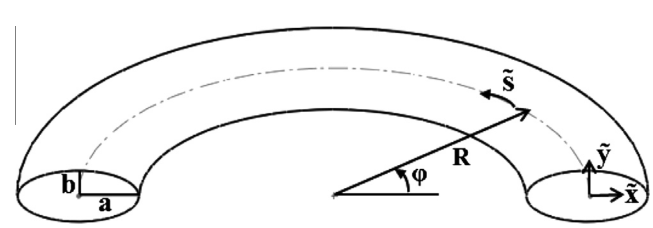

Fig. 1. Geometry of the problem.