Journal of Non-Newtonian Fluid Mechanics 216 (2015) 13–21

Contents lists available at ScienceDirect

Journal of Non-Newtonian Fluid Mechanics

journal homepage: http://www.elsevier.com/locate/jnnfm

Tensorial implicit constitutive relations in mechanics of incompressible non-Newtonian fluids

Tereza Perlácová, Vít Průša *

Faculty of Mathematics and Physics, Charles University in Prague, Sokolovská 83, Praha 8 – Karlín CZ 186 75, Czech Republic

ARTICLE INFO

Article history:
Received 7 July 2014
Received in revised form 8 December 2014
Accepted 12 December 2014
Available online 23 December 2014

Keywords:
Implicit constitutive relations
Experimental data
Normal stress differences
Conformation hysteresis

ABSTRACT

The standard assumption in the phenomenological theory of constitutive relations for non-Newtonian fluids is that the Cauchy stress tensor is a function of the symmetric part of the velocity gradient. By discussing experimental data available in the literature we show that the classical framework is overly restrictive. A simple framework that goes beyond the standard approach is the novel concept of implicit constitutive relations. Here, the basic assumption is that the relation between the stress and the symmetric part of the velocity gradient is given by an implicit tensorial equation. We demonstrate that the implicit type constitutive relations are adequate for fitting the one dimensional (shear stress versus shear rate) experimental data, and we speculate about the possible form of the corresponding three dimensional (Cauchy stress tensor versus symmetric part of the velocity gradient) implicit constitutive relations. Using the representation theorem for isotropic tensorial functions we conjecture that the implicit constitutive relations could lead to novel models capable to describe nonzero normal stress differences. Finally, we provide an example of a nontrivial thermodynamically and dynamically admissible implicit type tensorial constitutive relation. The simple model does predict nonzero normal stress difference, and shows that the conjecture is correct.

1. Introduction

The key step in the modelling of the behaviour of a non-Newtonian fluid is the specification of the constitutive relation which determines the response of the fluid to the given stimuli. In the simplest purely mechanical setting the specification of the constitutive relation requires to find a formula for the stress tensor in terms of the kinematical variables. This task can be accomplished using various approaches, see for example Tanner and Walters [1]. One option is to infer the macroscopic properties of the fluid from its microscopic structure, see for example Bird et al. [2]. The other option is the phenomenological approach. Here the idea is to work exclusively with macroscopic quantities, and use macroscopic concepts such as frame indifference or restrictions following from the laws of thermodynamics in order to specify the constitutive relation.

* Corresponding author.
E-mail addresses: teri.perlacova@gmail.com (T. Perlácová), prusv@karlin.mff.cuni.cz (V. Průša).

1.1. Standard algebraic constitutive relations

The standard phenomenological type approach to the so-called *algebraic models* goes as follows. The standard Navier–Stokes fluid model for an isotropic incompressible homogeneous fluid reads $\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D}$, where $\mathbb{T}$ denotes the Cauchy stress tensor, p is the pressure, $\mathbb{D}$ denotes the symmetric part of the velocity gradient, and μ, the viscosity, is a constant. If we denote $\mathbb{T}_\delta$ the traceless part of the Cauchy stress tensor, then the constitutive relation for the Navier–Stokes fluid can be, in virtue of the incompressibility constraint $\operatorname{div} \mathbf{v} = 0$, rewritten as

$$\mathbb{T}_\delta = 2\mu\mathbb{D}. \tag{1.1}$$

Since we need a generalisation of the constitutive relation (1.1), it is natural to assume that (1.1) should be replaced by

$$\mathbb{T}_\delta = \mathfrak{f}(\mathbb{D}), \tag{1.2}$$

where $\mathfrak{f}$ is a tensorial function. In such a case one talks about the standard *algebraic models* for incompressible homogeneous non-Newtonian fluids. Here we use the term *algebraic models* in order to emphasise that the constitutive relation does not contain higher spatial or time derivatives of the involved quantities such as in the case of the rate type or the differential type models. A further

http://dx.doi.org/10.1016/j.jnnfm.2014.12.006
0377-0257/

generalisation of (1.1) would be based on replacing the tensorial function $\mathfrak{f}$ in (1.2) by a functional, which leads directly to the celebrated notion of a simple fluid, see Noll [3].

1.2. Implicit algebraic constitutive relations

The theoretical phenomenological approach based on the notion of a simple fluid has not been fully followed in various works, see for example Seely [4], Blatter [5], Matsuhisa and Bird [6], where the generalised viscosity has been considered to be a function of the first invariant of the Cauchy stress. (See also Morgan [7] for an early discussion of implicit constitutive relations for the solids.) Despite this fact the concept of the simple fluid has played a dominant role in the phenomenological theory of constitutive relations.

More recently Rajagopal [8,9] has strongly argued that the standard theoretical phenomenological framework (1.2) is overly restrictive and urged for systematic rethinking of the standard approach. According to Rajagopal [9] one should instead of (1.2) rather consider *implicit algebraic constitutive relations*

$$\mathfrak{g}(\mathbb{T}_\delta, \mathbb{D}) = \mathbb{0}, \tag{1.3}$$

where $\mathfrak{g}$ is a tensorial function. A further generalisation would be based on replacing the tensorial function $\mathfrak{g}$ by a functional, see Průša and Rajagopal [10] for details.[1]

At first sight, using (1.3) instead of (1.2) in the theory of constitutive relations for non-Newtonian fluids seems to be a valueless theoretical triviality. However, the opposite is true. Replacing (1.2) by (1.3) provides a significant change of perspective in modelling of fluid like materials, and it has important consequences.

For example, constitutive relations for incompressible fluids with pressure dependent viscosity and Bingham type fluids do not fit into the standard framework (1.2). In both cases the traceless part of the Cauchy stress tensor $\mathbb{T}_\delta$ is not a *function* of the symmetric part of the velocity gradient $\mathbb{D}$. Using relations of type (1.3) offers a possibility to overcome this issue without the need to exploit complicated concepts such as multivalued mappings. Moreover, using the new framework it is possible to provide a solid thermodynamical background to these models, see Rajagopal [9], Rajagopal and Srinivasa [11] for the detailed discussion. Apart from providing an elegant theoretical approach to the Bingham type fluids and fluids with pressure dependent viscosity, the implicit type approach have been also used to give interesting insight into the formal theory of constitutive relations, see for example Průša and Rajagopal [10], Rajagopal [12].

Some specific examples of constitutive relations that fall into class (1.3), but do not belong to the class (1.2) have been discussed, on a theoretical basis, by Málek et al. [13], Le Roux and Rajagopal [14]. In particular, the referred authors have considered constitutive relations of type

$$\mathbb{D} = f(\mathbb{T}_\delta)\mathbb{T}_\delta, \tag{1.4}$$

where f is a scalar valued function. However, neither Málek et al. [13] nor Le Roux and Rajagopal [14] provide any experimental data that would justify the need to study constitutive relations of type (1.4).

Although the change of perspective from the standard setting to the implicit type setting has been already shown beneficial in the description of the response of some solid like[2] materials, see for example Freed and Einstein [15], Freed et al. [16] and Freed [17], studies concerning the potential of the implicit type constitutive relations in interpreting real experimental data for fluids are largely missing. The aim of the present work is to provide such a study. In particular we discuss the implications of the *algebraic* implicit constitutive relations (1.3) with respect to the interpretation of experimental data for some non-Newtonian fluids.

1.3. Fitting experimental data with tensorial implicit algebraic constitutive relations and related issues

In Sections 2 and 3, we show that the implicit type relations can be used to effectively fit experimentally observed data. Since the reported shear stress versus shear rate graphs have the characteristic S-shape, it is clear that *the experimental data cannot be in principle interpreted using the standard constitutive relations* of type (1.2). The only way how to interpret the data using an algebraic type relation between the stress and kinematical variables is to use the implicit type approach (1.4) or (1.3). This finding contributes to the arguments by Rajagopal [8,9] by pointing out new experimentally based constitutive relations that fit into framework (1.3), and that go beyond the simple examples discussed by Rajagopal [8,9].

Further, in the light of (1.3), we discuss theoretical issues related to the experimental procedures for the measurement of material parameters. In particular, we focus on the *problem of determining the fully three dimensional constitutive relation* (Cauchy stress tensor versus symmetric part of the velocity gradient) *from one dimensional measurements* (shear stress versus shear rate). The task of determining the three dimensional constitutive relation goes beyond simple quantitative curve fitting, and it is closely related to the qualitative issue of the presence of the normal stress differences. *We therefore discuss*, see Section 3.2, *the possibility of modelling the normal stress differences effect using the implicit constitutive relations*. We show that the algebraic implicit constitutive relations could lead to novel models capable to describe nonzero normal stress differences.

Finally, see Section 4, we provide an example of a nontrivial *thermodynamically and dynamically admissible implicit type tensorial constitutive relation*. The example explicitly shows that the alternative approach is viable from the dynamical and thermodynamical point of view.

2. One dimensional experimental data

2.1. Notation

Most of the experiments focused on the determination of the response of fluid like materials are based on measurements of the shear stress and the shear rate in the shear flow setting. Let us, mainly due to the need to fix the notation, recall the basic setup, see Fig. 1. The fluid is placed in between two parallel plates. The top plate moves with velocity V_{top} in the direction $\mathbf{e}_{\hat{z}}$, while the bottom plate is at rest.

The velocity field in the fluid is assumed to be a unidirectional velocity field $\mathbf{v} =_{\text{def}} v^{\hat{z}}(y)\mathbf{e}_{\hat{z}}$, and the Cauchy stress tensor is assumed to have the form

$$\mathbb{T} =_{\text{def}} \begin{bmatrix} \mathrm{T}_{\hat{x}\hat{x}} & 0 & 0 \\ 0 & \mathrm{T}_{\hat{y}\hat{y}} & \mathrm{T}_{\hat{y}\hat{z}} \\ 0 & \mathrm{T}_{\hat{z}\hat{y}} & \mathrm{T}_{\hat{z}\hat{z}} \end{bmatrix}, \tag{2.1}$$

where the elements of the stress tensor are assumed to be functions of y coordinate only. (Note that in virtue of the balance of angular momentum in absence of internal couples we know that $\mathrm{T}_{\hat{y}\hat{z}} = \mathrm{T}_{\hat{z}\hat{y}}$.) These are the natural assumptions that follow from the symmetry of the problem.

[1] If one wants to go beyond the phenomenological approach, then the constitutive relation replacing (1.3) could be the system $\mathfrak{g}(\mathbb{T}_\delta, \xi) = \mathbb{0}$, $\dot{\xi} = f(\mathbb{D}, \xi)$. Here ξ denotes a quantity characterising the internal microscopic structure of the material, and the dot denotes the time derivative. The system with an extra microscopic variable ξ could be seen as a more general and from microscopic point of view more physically meaningful alternative to (1.3). We thank the reviewer for pointing out this issue.

[2] The change of perspective that leads from (1.2) to (1.3) can be put into effect also in the theory of constitutive relations for solids. Instead of Cauchy stress tensor $\mathbb{T}$ and the symmetric part of the velocity gradient $\mathbb{D}$ one however needs to talk about, say, second Piola–Kirchhoff stress tensor $\mathbb{S}_R$ and Green strain tensor $\mathbb{E}$.