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Flow of a Burgers fluid due to time varying loads on deforming boundaries

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ABSTRACT

In this paper we study three boundary-initial value problems within the context of four rate type viscoelastic constitutive models, the Maxwell model, the Oldroyd-B model, Burgers model and the generalized Burgers model. We consider challenging problems wherein the boundary is deforming with time. The flows lead to a complex system of partial differential equations that require the development of a robust numerical procedure based on the arbitrary Lagrangian–Eulerian method.

1. Introduction

The response of many viscoelastic fluids can be captured reasonably well by rate type fluid models. The earliest and widely used rate type fluid model is that due to Maxwell [1] who developed a one dimensional model. Later, Burgers [2] developed a one dimensional rate type fluid model which includes the Maxwell model as a special case. A proper framework to develop frame indifferent three dimensional models was put into place by Oldroyd [3] and amongst the several models that he proposed one that has become very popular is the Oldroyd-B model. Burgers' one dimensional model includes the one dimensional Oldroyd-B model as a special case. These rate type models have been used to describe the behavior of a wide spectrum of materials: dilute polymeric fluids, asphalt and asphalt concrete, biological fluids, volcanic lava, glaciers, etc.

In view of their usefulness in describing the response of a wide variety of materials, many boundary-initial value problems have been solved within the context of these fluids. Usually the Maxwell and Oldroyd-B models are used in the simulations and we mention a few of them. For example, in a recent study, Damanik [4] in his Ph.D. thesis simulated the flows of Oldroyd-B and Giesekus model in a fixed domain in an Eulerian framework with FEM, he dealt with flows at high Weissenberg number wherein one encounters numerical difficulties that is referred to as the "High Weissenberg number Problem" (for more details about this problem see for example [5,6] or [7]) which is for example studied by Fattal and Kupferman using LCR reformulation in [8]. One of the problems he studied was the benchmark problem of a planar flow of the Oldroyd-B fluid around the cylinder where the drag force is computed, this problem has been studied numerically in many papers using both the finite element method and the finite volume method (see for example [6,9,10] or the paper by [11]).

The problem of free-surface flow was studied by Étienne et al. [12]; they studied the collapse of a column of Oldroyd-B fluid with the help of arbitrary Lagrangian–Eulerian formulation for low/moderate Weissenberg numbers. An alternative approach for a free surface problem is used in Damanik et al. [13] where the level set method is used for interface tracking between the viscoelastic bubble and the surrounding fluid. This approach has the capability to capture topological changes of the interface. We are also interested in problems concerning flows of asphalt involving free and deforming surfaces where one does not expect topological changes of the interface. In general these materials have been notoriously difficult to model (see the review article of Krishnan and Rajagopal [14] for a discussion of the relevant issues) and the popular model of choice for such materials is the model due to Burgers [2]. For the applications that we have in mind such as the compaction of asphalt layers, etc., the flow takes place at low to moderate Weissenberg numbers

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and hence we shall study such flows. The most popular model for describing the earth's mantle is also the Burgers fluid model, and as the model includes the Oldroyd-B and Maxwell models as special sub-classes our study is relevant to a very large class of problems wherein the boundary is undergoing time-dependent deformations. We solve problems with a deforming free surface by transforming the equations from Eulerian description into arbitrary Langrangian–Eulerian description. The way in which the equations are transformed is very general and can be easily used for implementing every viscoelastic rate type fluid model, even non-linear models. We are mainly interested in simulating Burgers' model which has not been so far simulated when free surface is deforming and which is capable of describing the response of material with more than one relaxation mechanism.

The problems with the viscoelastic fluid model under consideration is quite complicated and intricate as the constitutive relation is given by an implicit equation that relates the stress and properly invariant time derivatives of the stress and the symmetric part of the velocity gradient and its properly invariant time derivatives. Thus, unlike the classical theories of fluids such as the Euler fluid or the Navier–Stokes fluid, or constitutive theories wherein one has an explicit expression for the stress in terms of kinematical variable, which will allow us to substitute the expression for the same in the balance of linear momentum to obtain a partial differential equation for the velocity field, we will have to solve the system of equations comprising the constitutive equations and the basic balance laws simultaneously. The equations governing the flow of such fluids in general three dimensional problems in finite domains are most challenging and in this paper we consider three such problems that have relevance to interesting real world applications.

We consider three typical boundary-initial value problems. The first problem that we consider is a block of viscoelastic material that is initially at rest being subject to a compressive load on a part of the top surface of the block at time $t = 0$, and the compressive load is removed after application for a certain time. We then study the evolution of the deformation of the slab with time. This problem would correspond to a static load such as a parked vehicle. The second problem concerns a generalization of the first wherein we consider repeated application of compressive loads at two different locations on the top surface of the slab. This situation would correspond to the important technological problem of rutting of roadways, wherein a depression is observed in a portion of the roadways due to the repeated motion of vehicles. In the last problem, we consider a load that is moving on the top surface. This problem is relevant to the rolling of asphalt due to a roller, when the roadway is being built, or due to a moving vehicle.

The organization of the paper is as follows. In the next section, we introduce the Burgers model and a generalization of it. These models contain as special sub-classes two rate type models that are capable of describing the response of viscoelastic fluids: the Maxwell model and the Oldroyd-B model, and generalizations of the same. In Section 3 we discuss the numerical procedure, see for example the book by Crochet et al. [15] for relevant background. After a brief discussion of the unsuitability of the Lagrangian method to study the problem, we discuss the Arbitrary Lagrangian–Eulerian method that is used to simulate the three boundary-initial value problems discussed above. In the final section, we discuss the solution to three boundary-initial value problems.

2. Some standard incompressible viscoelastic rate type fluid models

In this section we introduce the incompressible viscoelastic rate-type fluid models that are used in the simulations due to time varying loads on the boundary. Since we consider incompressible fluids, they can only undergo isochoric motions and hence

$$\operatorname{div} \mathbf{v} = 0, \tag{1}$$

where $\mathbf{v}$ is the fluid velocity. The balance of linear momentum is in the form

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\nabla \mathbf{v})\mathbf{v}\right) = \operatorname{div} \mathbf{T}, \tag{2}$$

where ρ is the density and $\mathbf{T}$ is Cauchy stress tensor which is symmetric due to the balance of angular momentum in the absence of internal body couples.

Maxwell model. Maxwell [1] derived the earliest one dimensional fluid model which when appropriately generalized to three dimensions, in the case of an incompressible fluid, can be written in the form:

$$\mathbf{T} = -p\mathbf{I} + G(\mathbf{B} - \mathbf{I}), \tag{3a}$$

$$\overset{\triangledown}{\mathbf{B}} = \frac{1}{\tau}(\mathbf{I} - \mathbf{B}), \tag{3b}$$

where instead G is elastic modulus, τ is the relaxation time and $\triangledown$ is the upper convected Oldroyd derivative defined through

$$\overset{\triangledown}{\mathbf{B}} = \frac{\partial \mathbf{B}}{\partial t} + (\nabla \mathbf{B})\mathbf{v} - (\nabla \mathbf{v})\mathbf{B} - \mathbf{B}(\nabla \mathbf{v})^{\mathrm{T}}, \tag{4}$$

where $(\nabla \mathbf{v})_{ij} = \frac{\partial \mathbf{v}_i}{\partial x_j}$.

Oldroyd-B model. Oldroyd-B model was derived by Oldroyd [3], compared to Maxwell model the Cauchy stress tensor $\mathbf{T}$ is in the form

$$\mathbf{T} = -p\mathbf{I} + 2\eta_s\mathbf{D} + G(\mathbf{B} - \mathbf{I}), \tag{5}$$

where η_s is the solvent viscosity and $\mathbf{B}$ satisfies (3b).

Burgers model. Before Oldroyd, Burgers [2] had developed a one dimensional rate type constitutive relation which when properly generalized to three dimensions can be expressed as

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \tag{6a}$$

$$\mathbf{S} + \lambda_1 \overset{\triangledown}{\mathbf{S}} + \lambda_2 \overset{\triangledown\triangledown}{\mathbf{S}} = \eta_1\mathbf{D} + \eta_2 \overset{\triangledown}{\mathbf{D}}, \tag{6b}$$

where $\lambda_1, \lambda_2, \eta_1$ and η_2 are material parameters. This model includes both the Maxwell models and the Oldroyd-B models as special sub-classes. The three dimensional Burgers model can be expressed as a multimode (2-mode) Maxwell model (see [14]):

$$\mathbf{T} = -p\mathbf{I} + G_1(\mathbf{B}_1 - \mathbf{I}) + G_2(\mathbf{B}_2 - \mathbf{I}), \tag{7a}$$

$$\overset{\triangledown}{\mathbf{B}}_1 = \frac{1}{\tau_1}(\mathbf{I} - \mathbf{B}_1), \tag{7b}$$

$$\overset{\triangledown}{\mathbf{B}}_2 = \frac{1}{\tau_2}(\mathbf{I} - \mathbf{B}_2), \tag{7c}$$

where G_1, G_2 are elastic moduli and τ_1, τ_2 are relaxation times and so this model is capable of capturing two different relaxation mechanisms (compared to Maxwell or Oldroyd that are capable of capturing only one).

A modified Burgers model with additional Newtonian dissipation. We will use also the Burgers model with additional Newtonian dissipation

$$\mathbf{T} = -p\mathbf{I} + 2\eta_s\mathbf{D} + G_1(\mathbf{B}_1 - \mathbf{I}) + G_2(\mathbf{B}_2 - \mathbf{I}), \tag{8a}$$

$$\overset{\triangledown}{\mathbf{B}}_1 = \frac{1}{\tau_1}(\mathbf{I} - \mathbf{B}_1), \tag{8b}$$

$$\overset{\triangledown}{\mathbf{B}}_2 = \frac{1}{\tau_2}(\mathbf{I} - \mathbf{B}_2). \tag{8c}$$

One notices that Oldroyd-B model (5) reduces to a Maxwell model (3) when $\eta_s = 0$ Pa s, the same holds for the generalized Burgers model (8) and Burgers model (7). Further, Burgers (7) reduces to Maxwell model (3) if $\tau_1 = \tau_2 = \tau$ and $G = G_1 + G_2$ if we have the same boundary and initial conditions for $\mathbf{B}_1$ and $\mathbf{B}_2$.