Journal of Non-Newtonian Fluid Mechanics 213 (2014) 73–81

Contents lists available at ScienceDirect

Journal of Non-Newtonian Fluid Mechanics

journal homepage: http://www.elsevier.com/locate/jnnfm

The neutral curve for stationary disturbances in rotating disk flow for power-law fluids

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ARTICLE INFO

Article history:
Received 1 July 2014
Received in revised form 12 September 2014
Accepted 17 September 2014
Available online 28 September 2014

Keywords:
Instability
Rotating disk flow
Power-law fluid
Neutral curve

ABSTRACT

This paper is concerned with the convective instabilities associated with the boundary-layer flow due to a rotating disk. Shear-thinning fluids that adhere to the power-law relationship are considered. The neutral curves are computed using a sixth-order system of linear stability equations which include the effects of streamline curvature, Coriolis force and the non-Newtonian viscosity model. Akin to previous Newtonian studies it is found that the neutral curves have two critical values, these are associated with the type I upper-branch (cross-flow) and type II lower-branch (streamline curvature) modes. Our results indicate that an increase in shear-thinning has a stabilising effect on both the type I and II modes, in terms of the critical Reynolds number and growth rate. Favourable agreement is obtained between existing asymptotic predictions and the numerical results presented here.

1. Introduction

The stability and transition of the boundary-layer flow due to a rotating disk has attracted considerable interest in recent decades and continues to be an area of flourishing research. The pioneering study of Gregory et al. [1] contains the first observation of the stationary cross-flow vortices on a rotating disk. These instabilities were explained theoretically using a high Reynolds number linear stability analysis. Malik [2] presents the first comprehensive numerical study concerning the convective stationary disturbances, computing the curves of neutral stability using a sixth-order system of linear disturbance equations. Utilising a parallel-flow approximation as well as including the effects of streamline curvature and Coriolis force Malik demonstrates that there exists two distinct neutral branches. An upper-branch due to the cross-flow instability, termed type I and a lower-branch attributed to external streamline curvature, termed type II. These numerical results were verified by the linear asymptotic analysis of Hall [3]. He recovered the type I solution presented by Gregory et al. [1] (later corrected by Gajjar [4]) and showed that an additional short-wavelength mode exists, its structure being fixed by a balance between viscous and Coriolis forces. This mode corresponds directly to the type II branch.

Lingwood [5] investigated the role of absolute instability showing that the boundary-layer on a rotating disk of infinite extent is locally absolutely unstable at Reynolds numbers in excess of a critical value. The value of the critical Reynolds number agrees exceptionally well with experimental data, leading to Lingwood's hypothesis that absolute instability plays a part in turbulent transition on the rotating disk. Subsequently, Davies and Carpenter [6] investigated the global behaviour of the absolute instability of the rotating disk boundary-layer. By direct numerical simulations of the linearised governing equations they were able to show that the local absolute instability does not produce a linear global instability, instead suggesting that convective behaviour eventually dominates at all the Reynolds numbers. Their conclusion was that absolute instability was not involved in the transition process through linear effects. More recently, Pier [7] demonstrated explicitly that a non-linear approach is required to explain the self-sustained behaviour of the rotating disk flow. Using the result of Huerre and Monkewitz [8] that the presence of local absolute instability does not necessarily give rise to linear global instability; Pier suggested that the flow has a primary non-linear global mode (fixed by the onset of the local absolute instability) which has a secondary absolute instability that triggers the transition to turbulence.

Extending the rotating disk theory Lingwood [9], Lingwood and Garrett [10] considered the BEK system of rotating boundary-layer flows, named as such as it encompasses a family of rotating flows

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http://dx.doi.org/10.1016/j.jnnfm.2014.09.009

including the Bödewadt, Ekman and von Kármán boundary layers. They show that as the Rossby number increases the flows become increasingly unstable in both the convective and absolute senses. Noting that the onset of convective and absolute instability occurs almost simultaneously at very low Reynolds number in the Bödewadt boundary-layer. Numerous other studies have utilised and modified the numerical scheme employed by Lingwood [5]. Garrett and Peake [11] consider the stability and transition of the boundary-layer on a rotating sphere whilst Garrett et al. [12] investigate the cross-flow instability of the boundary-layer on a rotating cone. One particularly interesting extension, with respect to the von Kármán boundary-layer, is the temperature-dependent viscosity study of Jasmine and Gajjar [13]. The authors introduce a viscosity model based on an inverse linear function of temperature, controlled by the small parameter ϵ. They conclude that the stability of the flow is particularly sensitive to changes in viscosity and even for small positive values of ϵ the flow is much more unstable compared to the constant viscosity case defined by $\epsilon = 0$.

In the current paper we examine the linear convective instability of the boundary-layer on a rotating disk for power-law fluids. This work is essentially a development stemming from the linear asymptotic study of Griffiths et al. [14] who hypothesised that shear-thinning fluids may have a stabilising effect on the flow. Following the approach of Malik [2] we compute curves of neutral stability that can then be directly compared to the asymptotic predictions of Griffiths et al. [14]. A brief review of the inconsistencies regarding steady mean flow solutions for this problem is given by Griffiths et al. [14], and for the reasons outlined therein we restrict our attention to flows with moderate levels of shear-thinning. The interested reader is referred to Denier and Hewitt [15] for an in-depth analysis. In Section 2 the solution of the boundary-layer equations that give the steady mean flow profiles is described and the unsteady perturbation equations for the system are derived. The convective instability analysis is conducted in Section 3, where our theoretical predictions are compared with existing asymptotic results and linear convective growth rates are discussed. Finally, our conclusions are presented in Section 4.

2. Formulation

We consider the flow of a steady incompressible power-law fluid due to an infinite rotating plane located at $z^* = 0$. The plane rotates about the z^*-axis with angular velocity Ω^*. The motion of the fluid is in the positive z^* direction, the fluid is infinite in extent and the only boundary is located at $z^* = 0$. In a rotating frame of reference the continuity and Navier–Stokes equations are expressed as

$$\nabla \cdot \boldsymbol{u}^* = 0, \tag{1a}$$

$$\frac{\partial \boldsymbol{u}^*}{\partial t^*} + \boldsymbol{u}^* \cdot \nabla \boldsymbol{u}^* + \boldsymbol{\Omega}^* \times (\boldsymbol{\Omega}^* \times \boldsymbol{r}^*) + 2\boldsymbol{\Omega}^* \times \boldsymbol{u}^* = -\frac{1}{\rho^*}\nabla p^* + \frac{1}{\rho^*}\nabla \cdot \boldsymbol{\tau}^*. \tag{1b}$$

Here $\boldsymbol{u}^* = (\widetilde{U}^*, \widetilde{V}^*, \widetilde{W}^*)$ are the velocity components in cylindrical polar coordinates $(r^*, \theta, z^*), t^*$ is time, $\boldsymbol{\Omega}^* = (0,0,\Omega^*)$ and $\boldsymbol{r}^* = (r^*, 0, z^*)$. The fluid density is ρ^* and p^* is the fluid pressure. For generalised Newtonian models, such as the power-law model, the stress tensor is given by

$$\boldsymbol{\tau}^* = \mu^* \dot{\boldsymbol{\gamma}}^* \quad \text{with} \quad \mu^* = \mu^*(\dot{\gamma}^*),$$

where $\dot{\boldsymbol{\gamma}}^* = \nabla \boldsymbol{u}^* + (\nabla \boldsymbol{u}^*)^{\mathrm{T}}$ is the rate of strain tensor and $\mu^*(\dot{\gamma}^*)$ is the non-Newtonian viscosity. The magnitude of the rate of strain tensor is

$$\dot{\gamma}^* = \sqrt{\frac{\dot{\boldsymbol{\gamma}}^* : \dot{\boldsymbol{\gamma}}^*}{2}}.$$

The governing relationship for $\mu^*(\dot{\gamma}^*)$ when considering a power-law fluid is

$$\mu^*(\dot{\gamma}^*) = m^*(\dot{\gamma}^*)^{n-1}, \tag{2}$$

where m^* is the consistency coefficient and n is the dimensionless power-law index, with $n > 1, n < 1$ corresponding to shear-thickening and shear-thinning fluids, respectively. For $n = 1$ we recover the Newtonian viscosity model.

In the Newtonian limit an exact solution of (1) exists, as was first determined by von Kármán [16]. However, no such solution exists for flows with $n \neq 1$. It is only in the large Reynolds number limit that the leading order boundary-layer equations admit a similarity solution analogous to the exact Newtonian solution. The governing boundary-layer equations are

$$\frac{1}{r^*}\frac{\partial (r^* \widetilde{U}_0^*)}{\partial r^*} + \frac{1}{r^*}\frac{\partial \widetilde{V}_0^*}{\partial \theta} + \frac{\partial \widetilde{W}_0^*}{\partial z^*} = 0, \tag{3a}$$

$$\frac{\partial \widetilde{U}_0^*}{\partial t^*} + \widetilde{U}_0^*\frac{\partial \widetilde{U}_0^*}{\partial r^*} + \frac{\widetilde{V}_0^*}{r^*}\frac{\partial \widetilde{U}_0^*}{\partial \theta} + \widetilde{W}_0^*\frac{\partial \widetilde{U}_0^*}{\partial z^*} - \frac{(\widetilde{V}_0^* + r^*\Omega^*)^2}{r^*} = \frac{1}{\rho^*}\frac{\partial}{\partial z^*}\left(\tilde{\mu}^*\frac{\partial \widetilde{U}_0^*}{\partial z^*}\right), \tag{3b}$$

$$\frac{\partial \widetilde{V}_0^*}{\partial t^*} + \widetilde{U}_0^*\frac{\partial \widetilde{V}_0^*}{\partial r^*} + \frac{\widetilde{V}_0^*}{r^*}\frac{\partial \widetilde{V}_0^*}{\partial \theta} + \widetilde{W}_0^*\frac{\partial \widetilde{V}_0^*}{\partial z^*} + \frac{\widetilde{U}_0^*\widetilde{V}_0^*}{r^*} + 2\Omega^*\widetilde{U}_0^* = \frac{1}{\rho^*}\frac{\partial}{\partial z^*}\left(\tilde{\mu}^*\frac{\partial \widetilde{V}_0^*}{\partial z^*}\right), \tag{3c}$$

$$\frac{\partial \widetilde{W}_0^*}{\partial t^*} + \widetilde{U}_0^*\frac{\partial \widetilde{W}_0^*}{\partial r^*} + \frac{\widetilde{V}_0^*}{r^*}\frac{\partial \widetilde{W}_0^*}{\partial \theta} + \widetilde{W}_0^*\frac{\partial \widetilde{W}_0^*}{\partial z^*} = -\frac{1}{\rho^*}\frac{\partial \widetilde{P}_1^*}{\partial z^*} + \frac{1}{\rho^* r^*}\frac{\partial}{\partial r^*}\left(\tilde{\mu}^* r^*\frac{\partial \widetilde{U}_0^*}{\partial z^*}\right) + \frac{1}{\rho^* r^*}\frac{\partial}{\partial \theta}\left(\tilde{\mu}^*\frac{\partial \widetilde{V}_0^*}{\partial z^*}\right) + \frac{2}{\rho^*}\frac{\partial}{\partial z^*}\left(\tilde{\mu}^*\frac{\partial \widetilde{W}_0^*}{\partial z^*}\right), \tag{3d}$$

with the viscosity function $\tilde{\mu}^*$ given by

$$\tilde{\mu}^* = m^*\left[\left(\frac{\partial \widetilde{U}_0^*}{\partial z^*}\right)^2 + \left(\frac{\partial \widetilde{V}_0^*}{\partial z^*}\right)^2\right]^{(n-1)/2}. \tag{3e}$$

Here $(\widetilde{U}_0^*, \widetilde{V}_0^*, \widetilde{W}_0^*)$ are the leading order velocity components and $\widetilde{P}_1^*$ is the leading order fluid pressure term.

We introduce the generalisation of the classic Newtonian similarity solution in order to solve for the steady mean flow relative to the disk. The dimensionless similarity variables are defined by

$$U(\eta) = \frac{\widetilde{U}_0^*}{r^*\Omega^*}, \quad V(\eta) = \frac{\widetilde{V}_0^*}{r^*\Omega^*}, \quad W(\eta) = \frac{\widetilde{W}_0^*}{\chi^*}, \quad P(\eta) = \frac{\widetilde{P}_1^*}{\rho^*\chi^{*2}}, \tag{4}$$

where

$$\chi^* = \left[\frac{\nu^*}{r^{*1-n}\Omega^{*1-2n}}\right]^{1/(n+1)}.$$

Here (U, V, W) are the dimensionless radial, azimuthal and axial base flow velocities, respectively, P is the pressure and $\nu^* = m^*/\rho^*$ is the kinematic viscosity. The dimensionless similarity coordinate is

$$\eta = \frac{r^{*(1-n)/(n+1)}z^*}{L^{*2/(n+1)}} \quad \text{where} \quad L^* = \sqrt{\frac{\nu^*}{\Omega^{*2-n}}},$$