Composite Structures 142 (2016) 1–6

Contents lists available at ScienceDirect

Composite Structures

journal homepage: www.elsevier.com/locate/compstruct

On the closed form expression of the Mori–Tanaka theory prediction for the engineering constants of a unidirectional fiber-reinforced ply

Sergey G. Abaimov [a,*], Anastasia A. Khudyakova [a], Stepan V. Lomov [a,b]

[a] Center for Design, Manufacturing, and Materials, Skolkovo Institute of Science and Technology, Skolkovo, Russia
[b] Department of Materials Engineering, KU Leuven, Kasteelpark Arenberg 44, B-3001 Leuven, Belgium

A R T I C L E I N F O

Article history:
Available online 6 February 2016

Keywords:
Mori–Tanaka
Continuous unidirectional ply
Engineering constants
Transverse modulus

A B S T R A C T

The Mori–Tanaka theory is well-known as one of the most accurate approximations of mechanical properties of composite materials in structural analysis. However, while the closed form expressions of its predictions for elastic stiffness constants are available, so far it has lacked similar expressions for the engineering constants typically required in applied engineering structural analysis. In this study, we provide a closed form expression of the Mori–Tanaka theory prediction for the engineering constants of a unidirectional fiber-reinforced ply including the expression for the transverse modulus.

1. Introduction

Many studies have investigated the prediction of the unidirectional fiber-reinforced ply stiffness properties on the base of empirical or statistical micromechanical models [1,2]. In particular, predictions of engineering constants are of special importance due to the demand of them in applied engineering structural analysis. A comparison of different available formulae can be found in [3–7].

While the theories in general agree on the prediction of the longitudinal modulus of the material, the transverse modulus presents a difficult question. The most widely applied approaches include the Chamis hypothesis [8] and the modified rule of mixtures [9]. These two models are unique in the sense that they provide closed form, compact expressions suitable for engineering purposes. On the contrary, other theories including Halpin–Tsai semiempirical expressions [10], the composite sphere and cylinder assemblage models [11–14], the three-phase model [14], the Mori–Tanaka theory [15,16], the self-consistent method [17–20], and the differential scheme [21–28] are more accurate but either not able to provide closed form expressions for all moduli, or these expressions are lengthy and, therefore, have limited applicability. In particular, for the Mori–Tanaka theory it is often said that although the calculations are direct, the analytical results for the transverse modulus are too lengthy to be published or employed in applied structural analysis. Expressions for Mori–Tanaka predictions are given in [29,30], but these formulae involve complex matrix operations and knowledge of Eshelby tensor. The summary of the available so far closed form expressions for the Mori–Tanaka theory can be found in [31] but these expressions do not include engineering constants.

The authors strongly believe in the usefulness of closed form expressions for the engineering constants of a material. Although in science the necessity for such expressions (when Hill's constants are available) may sometimes be limited, in industry such expressions are often in great demand due to their simplicity and convenience.

In this study, we overcome this drawback publishing the engineering constants of the unidirectional fiber-reinforced ply in closed form. The presented results are compact but exact (as much as the Mori–Tanaka approximation is valid). For the detailed comparisons of the values of effective engineering constants obtained by means of the Mori–Tanaka method with the results of other theories, we refer the Reader to reviews [3–5,7], to Section 5 of our paper, and to study [6] which presents the comparison of the finite-element homogenization for a unit cell of unidirectional composite with regular and random placement of fibers.

In Section 2, we discuss the properties of the composite constituents used to illustrate the formulae applications. In Section 3, we list the tensors used in homogenization as well as the expressions leading to the effective engineering constants of the material. In Section 4, we apply the Mori–Tanaka approximation to the unidirectional fiber-reinforced ply and find the effective engineering constants of the homogenized material. In Section 5, we present numerical comparison of the results for different composite constituents, considered earlier in Section 2.

* Corresponding author.
E-mail address: s.abaimov@skoltech.ru (S.G. Abaimov).

http://dx.doi.org/10.1016/j.compstruct.2016.02.001
0263-8223/

2. Materials considered

For illustrative purposes, we discuss the homogenization procedure applied to several particular composites. Mechanical properties (engineering constants) of the considered fibers and matrix as composite phases are given in Table 1, where $E_{f,1}$, $E_{f,2}$, $\nu_{f,12}$, $\nu_{f,23}$, $G_{f,12}$, and $G_{f,23}$ are the engineering constants (Young's moduli, Poisson's ratios, and shear moduli) of the fiber in its coordinate system (axis 1 is the fiber axis) and E_m, ν_m, and G_m are the engineering constants of the isotropic matrix.

Thereby, we consider the matrix and glass fibers as being isotropic. We approximate the carbon and flax fibers as being transversely isotropic.

3. Utilized tensors

Let us consider a fiber. In its coordinate system $(x_{f,1}, x_{f,2}, x_{f,3})$, we assume the stiffness properties of this fiber to be orthotropic with the compliance tensor

$$M_{f,1111} = \frac{1}{E_{f,1}}, \tag{1}$$

$$M_{f,2222} = M_{f,3333} = \frac{1}{E_{f,2}}, \tag{2}$$

$$M_{f,1122} = M_{f,1133} = -\frac{\nu_{f,12}}{E_{f,1}}, \tag{3}$$

$$M_{f,2233} = -\frac{\nu_{f,23}}{E_{f,2}}, \tag{4}$$

$$M_{f,2323} = \frac{1}{4G_{f,23}}, \tag{5}$$

$$M_{f,3131} = M_{f,1212} = \frac{1}{4G_{f,12}}. \tag{6}$$

All other elements of the compliance tensor, not listed above or not obtained from the listed above by implied symmetry relations $M_{f,ijkl} = M_{f,jikl} = M_{f,ijlk} = M_{f,klij}$, are zero.

The inverse of the compliance tensor constitutes the stiffness tensor

$$L_{f,1111} = \frac{E_{f,1}(1-\nu_{f,23})}{1-\nu_{f,23}-2\frac{E_{f,2}}{E_{f,1}}\nu_{f,12}^2}, \tag{7}$$

$$L_{f,2222} = L_{f,3333} = \frac{E_{f,2}\left(1-\frac{E_{f,2}}{E_{f,1}}\nu_{f,12}^2\right)}{(1+\nu_{f,23})\left\{1-\nu_{f,23}-2\frac{E_{f,2}}{E_{f,1}}\nu_{f,12}^2\right\}}, \tag{8}$$

$$L_{f,1122} = L_{f,1133} = \frac{E_{f,2}\nu_{f,12}}{1-\nu_{f,23}-2\frac{E_{f,2}}{E_{f,1}}\nu_{f,12}^2}, \tag{9}$$

$$L_{f,2233} = \frac{E_{f,2}\left(\nu_{f,23}+\frac{E_{f,2}}{E_{f,1}}\nu_{f,12}^2\right)}{(1+\nu_{f,23})\left\{1-\nu_{f,23}-2\frac{E_{f,2}}{E_{f,1}}\nu_{f,12}^2\right\}}, \tag{10}$$

$$L_{f,2323} = G_{f,23}, \tag{11}$$

$$L_{f,3131} = L_{f,1212} = G_{f,12}. \tag{12}$$

Again, all other elements of the tensor, not listed above or not obtained from the listed above by the implied symmetry relations $L_{f,ijkl} = L_{f,jikl} = L_{f,ijlk} = L_{f,klij}$, are zero.

Here we explicitly formulated that considered fibers are not required to be transversely isotropic in the sense that $G_{f,23} \neq \frac{E_{f,2}}{2(1+\nu_{f,23})}$. Therefore, the results obtained later will correspond to this general case of fiber structure and material.

For the isotropic matrix, the compliance and stiffness tensors are much simpler:

$$M_{m,1111} = M_{m,2222} = M_{m,3333} = \frac{1}{E_m}, \tag{13}$$

$$M_{m,1122} = M_{m,1133} = M_{m,2233} = -\frac{\nu_m}{E_m}, \tag{14}$$

$$M_{m,2323} = M_{m,3131} = M_{m,1212} = \frac{1+\nu_m}{2E_m}, \tag{15}$$

$$L_{m,1111} = L_{m,2222} = L_{m,3333} = \frac{E_m(1-\nu_m)}{1-\nu_m-2\nu_m^2}, \tag{16}$$

$$L_{m,1122} = L_{m,1133} = L_{m,2233} = \frac{E_m\nu_m}{1-\nu_m-2\nu_m^2}, \tag{17}$$

$$L_{m,2323} = L_{m,3131} = L_{m,1212} = \frac{E_m}{2(1+\nu_m)}. \tag{18}$$

If we homogenize the unidirectional fiber-reinforced ply, its properties are described by the definitions similar to (1–12) in the material coordinate system (x_1, x_2, x_3); only instead of fiber engineering constants $E_{f,1}$, $E_{f,2}$, $\nu_{f,12}$, $\nu_{f,23}$, $G_{f,12}$, and $G_{f,23}$, we should substitute the effective engineering constants of the material:

$$\left\langle M_{1111}^{ply}\right\rangle = \frac{1}{E_1^{eff}}, \tag{19}$$

$$\left\langle M_{2222}^{ply}\right\rangle = \left\langle M_{3333}^{ply}\right\rangle = \frac{1}{E_2^{eff}}, \tag{20}$$

$$\left\langle M_{1122}^{ply}\right\rangle = \left\langle M_{1133}^{ply}\right\rangle = -\frac{\nu_{12}^{eff}}{E_1^{eff}}, \tag{21}$$

$$\left\langle M_{2233}^{ply}\right\rangle = -\frac{\nu_{23}^{eff}}{E_2^{eff}}, \tag{22}$$

Table 1
Engineering constants of the considered composite phases.

Material	Young's modulus E, GPa		Poisson ratio		Shear modulus G, GPa	
HM Carbon P-100 fiber [32,33]	$E_{f,1} = 775$	$E_{f,2} = 6.8$	$\nu_{f,12} = 0.22$	$\nu_{f,23} = 0.28$	$G_{f,12} = 20.6$	$G_{f,23} = \frac{E_{f,2}}{2(1+\nu_{f,23})} = 2.7$
HS Carbon T650 fiber [4,32,33]	$E_{f,1} = 243$	$E_{f,2} = 13.8$	$\nu_{f,12} = 0.29$	$\nu_{f,23} = 0.28$	$G_{f,12} = 23.1$	$G_{f,23} = \frac{E_{f,2}}{2(1+\nu_{f,23})} = 5.4$
E-Glass fiber [34,35]	$E_f = 72$		$\nu_f = 0.22$		$G_f = \frac{E_f}{2(1+\nu_f)} = 29.5$	
Flax fiber [36,37]	$E_{f,1} = 60$	$E_{f,2} = 10$	$\nu_{f,12} = 0.25$	$\nu_{f,23} = 0.25$	$G_{f,12} = 3.2$	$G_{f,23} = \frac{E_{f,2}}{2(1+\nu_{f,23})} = 4.0$
Epoxy PMR-15 [4,38]	$E_m = 3.3$		$\nu_m = \frac{E_m}{2G_m} - 1 = 0.375$		$G_m = 1.2$	