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Boundary layers for the upper convected Maxwell fluid

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ABSTRACT

We discuss boundary layers arising in the high Weissenberg number limit of viscoelastic flows and the scalings that arise in analyzing them. Two quite distinct mechanisms for the formation of viscoelastic boundary layers exist: One mechanism is analogous to that of the Prandtl boundary layer and is linked to the enforcement of the no-slip boundary condition. The other mechanism has nothing to do with the no-slip condition, but is linked to long memory which manifests itself in different stresses near the wall versus some distance from the wall. In certain situations, these stress boundary layers may be embedded within Prandtl type boundary layers.

1. Introduction

The theory of viscous boundary layers in Newtonian fluids was developed by Prandtl [11] more than a century ago. Formally, the Navier–Stokes equations reduce to the Euler equations when the Reynolds number becomes infinite. However, the Euler equations do not allow the imposition of the no-slip boundary condition.

In his fundamental work [11], Prandtl made the conjecture that for many high Reynolds number flows, the Euler equations provide an adequate description except in a thin layer close to the boundary. By scaling analysis, he established a system of governing equations for this boundary layer, which are called Prandtl's equations.

Despite of the fact that many important applications can be based on the solution of Prandtl's system, its mathematical analysis is far from an easy problem. Oleinik and Samokhin [10] established a well-posedness result under the assumption that the velocity profile in the boundary layer is monotone. Sammartino and Caflisch [20] established an existence result for analytic initial data. We also refer to the review article of E [2] for further work prior to 2000. Recently, Gérard-Varet and Dormy [4] established that, for general initial data, the Prandtl equations are not well-posed in Sobolev spaces if the velocity profile is non-monotone. This result builds on a shear flow instability which was analyzed by Cowley et al. [1].

In this paper, we are concerned with viscoelastic flows governed by the upper convected Maxwell model. The high Weissenberg number limit of the equations governing this fluid can formally be reduced to a system of equations which is similar to ideal magnetohydrodynamics [9]. The well-posedness of this system was analyzed in [21]. Like the Euler equations, this limiting system allows only the imposition of the non-penetration condition, but not no-slip. As a result, a boundary layer forms, which is analogous to Prandtl's. In contrast to the Prandtl equations, the boundary layer equations for this situation were found to be well-posed [19].

Boundary layers like Prandtl's can arise in the time evolution of a flow or in "developing" flows like the well known Blasius solution for flow past an obstacle. Here, the boundary layer grows in thickness along the obstacle. However, it is counterintuitive to expect a sharp transition in velocity at a stationary boundary to persist in a fully developed steady flow. There is, however, an entirely different mechanism which leads to boundary layers in steady viscoelastic flows. In these flows, the velocity vanishes at the wall, even in the outer flow solution. High Weissenberg number means long relaxation time. Particles away from the wall therefore travel long distances within a relaxation time. On the other hand, particles close to the wall travel only a short distance. This leads to boundary layers in the stress.

Boundary layers of this latter type were analyzed in [14,16,17]. They play an essential role in the analysis of corner singularities [5,13,12,3], and they have long been observed in numerical simulations of steady viscoelastic flows [6,7].

The objective of this paper is to give an overview over the "zoo" of boundary layers that can arise in viscoelastic flows and discuss the mechanisms and scalings behind them. In Section 2, we discuss unsteady flows and boundary layers of the Prandtl type, which are due to a failure of the no-slip condition in the outer flow. In Section 3, we discuss the case of steady flow. We show why Prandtl type boundary layers on a stationary boundary

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cannot in general exist in a fully developed flow. We then discuss stress boundary layers and the possibility of having such stress boundary layers embedded as sublayers of a Prandtl boundary layer.

We consider the upper convected Maxwell model:

$$\begin{aligned}
&\rho\left(\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}\right)=\nabla\cdot\mathbf{T}-\nabla p,\\
&\nabla\cdot\mathbf{v}=0, \qquad\qquad (1)\\
&\tau\left(\frac{\partial \mathbf{T}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{T}-(\nabla\mathbf{v})\mathbf{T}-\mathbf{T}(\nabla\mathbf{v})^T\right)+\mathbf{T}=\eta(\nabla\mathbf{v}+(\nabla\mathbf{v})^T),
\end{aligned}$$

where $\mathbf{v}$, $\mathbf{T}$, p are velocity, stress and isotropic pressure while ρ, τ, η are the density, relaxation time and viscosity.

Assuming typical scales $L, U, \frac{L}{U}, \frac{\eta U}{L}$ for length, velocity, time and stress, we have the dimensionless system

$$\begin{aligned}
&\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\frac{1}{\mathbf{Re}}\nabla\cdot\mathbf{T}-\nabla p,\\
&\nabla\cdot\mathbf{v}=0,\\
&\frac{\partial \mathbf{T}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{T}-(\nabla\mathbf{v})\mathbf{T}-\mathbf{T}(\nabla\mathbf{v})^T+\frac{1}{\mathbf{Wi}}\mathbf{T}=\frac{1}{\mathbf{Wi}}(\nabla\mathbf{v}+(\nabla\mathbf{v})^T),
\end{aligned} \qquad (2)$$

where $\mathbf{Re} = \rho UL/\eta$ is the Reynolds number and $\mathbf{Wi} = \tau U/L$ is the Weissenberg number.

2. Boundary layers in unsteady flows

2.1. Outer problem

When the Weissenberg number $\mathbf{Wi}$ approaches infinity, a limiting set of equations arises which is analogous to the Euler equations in Newtonian flow. We note that, in the limit $\mathbf{Wi} = \infty$, the constitutive equation in (2) becomes invariant under a scaling of stresses. In a fully developed steady flow, we would expect viscometric flow at the wall, which leads to stresses of order $\mathbf{Wi}$. It would therefore be reasonable to expect stresses of a similar magnitude away from the wall as well. However, in this section we are concerned with an initial value problem. We note that for large $\mathbf{Wi}$ the time that is needed for stresses to reach their ultimate values is large of order $\mathbf{Wi}$. We shall therefore make no specific assumption on the magnitude of the initial stresses, and introduce a new dimensionless parameter which measures this magnitude. We call this parameter the Deborah number $\mathbf{De}$. In other words, $\mathbf{De}$ is the typical magnitude of the initial stresses relative to $\eta U/L$.

We now scale the stresses with an additional factor $\mathbf{De}$, to obtain

$$\begin{aligned}
&\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\frac{\mathbf{De}}{\mathbf{Re}}\nabla\cdot\mathbf{T}-\nabla p,\\
&\nabla\cdot\mathbf{v}=0,\\
&\frac{\partial \mathbf{T}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{T}-(\nabla\mathbf{v})\mathbf{T}-\mathbf{T}(\nabla\mathbf{v})^T+\frac{1}{\mathbf{Wi}}\mathbf{T}=\frac{1}{\mathbf{DeWi}}(\nabla\mathbf{v}+(\nabla\mathbf{v})^T).
\end{aligned} \qquad (3)$$

The elasticity number $E=\frac{\mathbf{De}}{\mathbf{Re}}$ plays a key role. We consider the interior problem in three cases, $E=\frac{\mathbf{De}}{\mathbf{Re}}$ is of order one, E is small and E is large.

2.1.1. $E=\frac{\mathbf{De}}{\mathbf{Re}}$ is of order one

In [21], we assume the elasticity constant $E=\frac{\mathbf{De}}{\mathbf{Re}}$ fixed with order one. Formally setting $\mathbf{Wi} = \infty$ above, we obtain the limiting system

$$\begin{aligned}
&\frac{\partial \mathbf{v}^0}{\partial t}+(\mathbf{v}^0\cdot\nabla)\mathbf{v}^0=E\nabla\cdot\mathbf{T}^0-\nabla p^0,\\
&\frac{\partial \mathbf{T}^0}{\partial t}+(\mathbf{v}^0\cdot\nabla)\mathbf{T}^0-(\nabla\mathbf{v}^0)\mathbf{T}^0-\mathbf{T}^0(\nabla\mathbf{v}^0)^T=\mathbf{0}, \qquad (4)\\
&\nabla\cdot\mathbf{v}^0=0.
\end{aligned}$$

The mathematical well-posedness of this system was studied in [21]. In doing so, we assume that $\mathbf{T}^0 \cdot \mathbf{n} = \mathbf{0}$ at the wall. The rationale for this is that the flow at the wall is necessarily a shear flow, and therefore we expect the stress to be dominated by the first normal stress component if $\mathbf{Wi}$ is large. It can be shown that, indeed, the condition $\mathbf{T}^0 \cdot \mathbf{n} = \mathbf{0}$ is preserved by the equations if it is satisfied by the initial data. An important consequence of this condition on the stress is that only the condition $\mathbf{v}^0 \cdot \mathbf{n} = 0$ can be imposed on the equations, but not the no-slip condition. This situation is analogous to that of the Euler equations.

2.1.2. $E=\frac{\mathbf{De}}{\mathbf{Re}}$ is small

When $E = 0$ in (4), the momentum equations decouple from the constitutive equations and become inviscid flow. Stress is decoupled, and can be easily recovered from a linear transport system.

2.1.3. $E=\frac{\mathbf{De}}{\mathbf{Re}}$ is large

When E becomes very large, we get a creeping flow,

$$\begin{aligned}
&\nabla\cdot\mathbf{T}^0=\nabla p^0,\\
&\nabla\cdot\mathbf{v}^0=0, \qquad (5)\\
&\frac{\partial \mathbf{T}^0}{\partial t}+(\mathbf{v}^0\cdot\nabla)\mathbf{T}^0-(\nabla\mathbf{v}^0)\mathbf{T}^0-\mathbf{T}^0(\nabla\mathbf{v}^0)^T=\mathbf{0}.
\end{aligned}$$

This creeping flow system does not lead to a well-posed initial value problem. For instance, we obtain a solution if we set $\mathbf{T}^0 = \mathbf{0}$, and take $\mathbf{v}^0$ to be a divergence free vector field with any time dependence whatsoever. The problem is that, to get a well-posed system for determining $\mathbf{v}^0$, we need $\mathbf{T}^0$ to be strictly positive definite. But we cannot expect the leading contribution to the stress to be positive definite in the limit of high $\mathbf{Wi}$. For more discussion, see [18]. For the rest of this section, we shall assume that E is at least of order one.

2.2. Boundary layer

Above we discuss the interior region at infinite Weissenberg number with boundary condition $\mathbf{v} \cdot \mathbf{n} = 0$. For the equations with finite $\mathbf{Wi}$, however, we are forced to impose the boundary condition $\mathbf{v} = \mathbf{0}$, not just $\mathbf{v} \cdot \mathbf{n} = 0$. To accommodate this, boundary layers must form near the wall. In analyzing these boundary layers, it is convenient to set $\mathbf{S} = \mathbf{T} + \mathbf{I}/\mathbf{DeWi}$. With this substitution, the constitutive law in (3) transforms to

$$\frac{\partial \mathbf{S}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{S}-(\nabla\mathbf{v})\mathbf{S}-\mathbf{S}(\nabla\mathbf{v})^T+\frac{1}{\mathbf{Wi}}\left(\mathbf{S}-\frac{1}{\mathbf{DeWi}}\mathbf{I}\right)=\mathbf{0}, \qquad (6)$$

and we have div $\mathbf{T}$ = div $\mathbf{S}$ in the momentum equation. The change in boundary conditions is related to the fact that, while $\mathbf{S} \cdot \mathbf{n}$ vanishes on the boundary at leading order, for the full equations we have strict positive definiteness of $\mathbf{S}$.

Now we study how to formulate the boundary layer system. We shall show that the boundary layer thickness is $1/\sqrt{\mathbf{WiRe}}$ when $\mathbf{Wi}$ is large. When $\mathbf{Wi}$ is small we recover the Prandtl system with a viscous boundary layer of thickness $1/\sqrt{\mathbf{Re}}$. For the formulation of boundary layer equations, we shall assume that the domain of flow is the half plane $y > 0$. This assumption is made only to simplify the exposition; the boundary layer equations remain valid