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Bounds on the natural frequencies of laminated rectangular plates with Extension–Bending coupling

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ABSTRACT

Extension–Bending coupling behaviour is believed to arise uniquely in anti-symmetric Cross-ply laminates. The results in this article serve to dispel this misconception by presenting solutions for both Cross-ply laminates and Standard laminates, defined as containing combinations of angle plies and cross plies, for a range of different sub-sequence symmetries. Details of the algorithm used to develop the definitive list of laminate stacking sequences with up to 21 plies are presented. Finally, natural frequency factor envelopes are developed, which identify significant differences in the bounds, across a range of aspect-ratios, between Standard laminate configurations and the previously assumed Cross-ply designs.

1. Introduction

The use of laminated composite materials continues to expand at pace beyond traditional aerospace applications, and into other fields of advanced technology. This expansion is due, in part, to a growing awareness of the unique and largely unexploited thermo-mechanical properties that are a potential enabling technology if correctly tailored.

Buckling strength and natural frequency are important indices in the assessment of new and advanced laminate designs, and closed form solutions, which aid preliminary design for compression buckling strength and natural frequency prediction of simply supported rectangular plates are well known and well documented in the literature for both isotropic and uncoupled laminated composite materials [1–3]. Less well known are closed form solutions for coupled laminates, derived previously for so called anti-symmetric Cross-ply laminates [4], possessing *Extension–Bending* coupling, and for anti-symmetric Angle-ply laminates, with *Extension–Twisting* and *Shearing–Bending* coupling [5,6]. This is perhaps due to widespread misunderstanding of coupled laminate behaviour, highlighted by Leissa [7], where many buckling results have been presented on the false assumption that bifurcation buckling can occur, when in fact simply supported rectangular plates consisting of Cross-ply laminates with *Extension–Bending* coupling will bend, and not buckle, when subject to in-plane compressive load. This message appears to have gone unheeded by some [8]. Only the natural frequency predictions remain valid for this class of coupled laminate. However, given that *Extension–Bending* coupled laminates also require either special curved tooling or cold cure resin systems to achieve the desired shape after manufacture and remain subject to thermal distortions in service, this renders them functional rather than structural in nature.

Misunderstanding of coupled laminates also extends to the widely held assumptions that *Extension–Bending* coupling is restricted to anti-symmetric, or more correctly, cross-symmetric Cross-ply laminates with even ply number groupings. Cross-ply laminates will be shown to represent only a subset of this class of coupled laminate; many configurations represent Standard laminates, defined as containing combinations of both cross-plies and angle-plies. This is an important discovery, particularly if the larger design space leads to a significant change in the bounds on the natural frequencies.

In this article therefore, definitive listings of laminate configurations with *Extension–Bending* coupling are derived for up to 21 plies, to complement previous studies on other laminate classes. All plies are assumed to possess identical fibre–matrix properties, e.g. carbon–epoxy or glass–epoxy, etc., with constant thickness throughout, differing only by their orientations. The listings comprise individual stacking sequences, separated into Cross-ply laminates, representing the generally assumed (anti-symmetric) form for this class of laminate, and Standard laminates, deemed to represent laminates containing cross ply and angle ply combinations with standard orientations, i.e., 90° and/or 0°, +45°, −45°. In

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both cases, no constraint is imposed on the form of sub-sequence symmetry.

The stacking sequences are given in symbolic form since the angle-plies (+/−) can in fact be assigned any arbitrary orientation, $0^\circ < \theta < 90^\circ$, and the cross-plies (○/●, representing 0°/90°) can be interchanged. Each stacking sequence is presented together with a set of non-dimensional parameters, derived in Section 2, from which the laminate stiffness properties can be readily determined for any fibre–matrix system.

Section 3 presents the relevant background theory and the derivation of the closed form solution used to calculate the natural frequency results that then follow. Section 4 presents the frequency bounds for both Standard and Cross-ply laminates. These comparisons involve the upper- and lower-bounds on the first three natural frequencies for *Extension–Bending* coupled laminates, over a broad range of aspect-ratios.

2. Extension–Bending coupled laminates

2.1. Laminate characterisation

Laminated composite materials have recently been characterised [9] in terms of their response to mechanical and/or thermal loading, to help understand coupling behaviour not present in conventional materials. Eq. (1) describes the well-known **ABD** relation from classical lamination theory, relating force {**N**} and moment {**M**} resultants with in-plane strains {**ε**} and curvatures {**κ**}, and from which the coupling behaviour is, by inspection, dependent on the form of the elements, A_{ij}, B_{ij} and D_{ij}, of the extensional [**A**], coupling [**B**] and bending [**D**] stiffness matrices, respectively:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$
$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{1}$$

The coupling behaviour can be described by a shorthand notation, using an extended subscript notation defined previously by the Engineering Sciences Data Unit [10]. Cross-symmetric Cross-ply laminates are referred to by the designation $\mathbf{A}_S\mathbf{B}_l\mathbf{D}_S$, signifying that the extensional stiffness matrix [**A**] is *Simple* in nature, i.e., uncoupled, since

$$A_{16} = A_{26} = 0 \tag{2}$$

the coupling matrix [$\mathbf{B}_l$] has non-zero leading-diagonal elements, i.e.,

$$B_{11},\ B_{22} \neq 0 \tag{3}$$

with all other elements zero and; the bending stiffness matrix [$\mathbf{D}_S$] is *Simple* in nature, i.e., uncoupled, since

$$D_{16} = D_{26} = 0 \tag{4}$$

Alternatively, coupled laminates may also be described in terms of the response that they exhibit under various combinations of force and moment resultants (or equally, the force and moment resultants that arise from enforced strains or curvatures), using a *cause–effect* relationship. Cross-symmetric Cross-ply laminates may therefore be described as *E–B* laminates, since *Extension* (*E*) causes a *Bending* (*B*) effect. Each *cause–effect* pair is reversible, e.g. *B–E*. Note that this response-based labelling is complementary to the Engineering Sciences Data Unit subscript notation [10].

2.2. Derivation of Extension–Bending coupled laminates

This section describes the derivation of two definitive listings of laminate stacking sequences and associated non-dimensional parameters, from which the stiffness properties are readily derived for any fibre/matrix system.

In the derivation of the stacking sequences for laminates with *Extension–Bending* coupling, it has been assumed that all plies have identical material properties and thickness, *t*, and differ only with respect to ply orientation.

The general rules of symmetry are relaxed. Stacking sequences are presented in symbolic form to allow for non-standard ply orientations, although, all natural frequency results presented relate to standard ply angle orientations, ±45°, 0° and 90°. For Standard laminates, all stacking sequences have an Angle-ply (+) on one surface (1st ply) of the laminate, but the other surface ply may have equal (+) or opposite (−) orientation or it may indeed be a cross ply (○ or ●) of 0° or 90° orientation. This constraint prevents the development of Cross-ply laminates, i.e., those containing cross plies only. For Cross-ply laminates, the constraint of one outer surface cross ply (○) is applied to avoid duplicate stacking sequences when cross plies are exchanged, 0° with 90° and vice versa.

Non-dimensional parameters are developed to allow for any fiber/matrix system. The derivation of non-dimensional stiffness parameters is readily demonstrated for the example of a 9-ply laminate, with anti-symmetric angle plies and non-symmetric cross plies: $[+/-/○/-/●/+/●/+/-]_T$, using the well-known stiffness relationships:

$$\begin{aligned} A_{ij} &= \sum Q'_{ij,k}(z_k - z_{k-1}) \\ B_{ij} &= \sum Q'_{ij,k}(z_k^2 - z_{k-1}^2)/2 \\ D_{ij} &= \sum Q'_{ij,k}(z_k^3 - z_{k-1}^3)/3 \end{aligned} \tag{5}$$

where the summation may instead be written in sequence order for the (k = 1, 2, ...,) 9 individual plies, and where *z*, representing the distance from the laminate mid-plane, is expressed here in terms of the uniform ply thickness, *t*, see Fig. 1. For the coupling stiffness matrix:

$$\begin{aligned} B_{ij} = \Big\{ & Q'_{ij,+45}((-7t/2)^2 - (-9t/2)^2) + Q'_{ij,-45}((-5t/2)^2 - (-7t/2)^2) \\ & + Q'_{ij,0}((-3t/2)^2 - (-5t/2)^2) + Q'_{ij,-45}((-t/2)^2 - (-3t/2)^2) \\ & + Q'_{ij,90}((t/2)^2 - (-t/2)^2) + Q'_{ij,+45}((3t/2)^2 - (t/2)^2) \\ & + Q'_{ij,90}((5t/2)^2 - (3t/2)^2) + Q'_{ij,+45}((7t/2)^2 - (5t/2)^2) \\ & + Q'_{ij,-45}((9t/2)^2 - (7t/2)^2) \Big\} /2 \end{aligned} \tag{6}$$

and the transformed reduced stiffness Q'_{ij}, with subscripts i, j = 1, 2, 6, corresponds to the orientation of ply *k*.

The coupling stiffness contributions for each ply orientation may be summarised as:

$$\begin{aligned} B_{ij,+45} &= \frac{0t^2}{2} \times Q'_{ij,+45} = \chi_{+45} t^2/4 \times Q'_{ij,+45} \\ B_{ij,-45} &= \frac{0t^2}{2} \times Q'_{ij,-45} = \chi_{-45} t^2/4 \times Q'_{ij,-45} \\ B_{ij,0} &= -\frac{4t^2}{2} \times Q'_{ij,0} = \chi_0 t^2/4 \times Q'_{ij,0} \\ B_{ij,90} &= \frac{4t^2}{2} \times Q'_{ij,90} = \chi_{90} t^2/4 \times Q'_{ij,90} \end{aligned} \tag{7}$$

where the non-dimensional parameters $\chi_0 = -8$, $\chi_{90} = 8$ and $\chi_{+45} = -\chi_{-45} = 0$.