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The shear-driven Rayleigh problem for generalised Newtonian fluids

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ABSTRACT

We consider a variant of the classical 'Rayleigh problem' ('Stokes's first problem') in which a semi-infinite region of initially quiescent fluid is mobilised by a shear stress applied suddenly to its boundary. We show that self-similar solutions for the fluid velocity are available for any generalised Newtonian fluid, regardless of its constitutive law. We demonstrate how these solutions may be used to provide insight into some generic questions about the behaviour of unsteady, non-Newtonian boundary layers, and in particular the effect of shear thinning or thickening on the thickness of a boundary layer.

1. Introduction: the Rayleigh problem

The Rayleigh problem, sometimes called Stokes's first problem, was first formulated as a note to the celebrated paper by Stokes [1], and later discussed more fully by Rayleigh [2]. The problem is to describe the behaviour of a semi-infinite region of fluid ($y < 0$ in the notation we will use), bounded by a plane wall at $y = 0$ and initially at rest, when the wall is impulsively accelerated to move in its own plane at a constant speed U.

For a Newtonian fluid, the momentum equation in the x-direction reduces to the linear diffusion equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{1}$$

where the velocity $\mathbf{u} = u(y,t)\mathbf{i}$ and ν is the constant kinematic viscosity. The velocity field then has a self-similar form (see e.g. Drazin and Riley [3, Section 4.2]),

$$u(y,t) = U(1 + \mathrm{erf}(\eta)), \quad \text{where} \quad \eta = \frac{y}{2(\nu t)^{1/2}}, \tag{2}$$

and where erf is the standard error function. Drazin and Riley [3] also note that by a simple change of variables the solution (2) may be used to describe the case in which, instead of being driven by a velocity applied suddenly at $y = 0$, the flow is driven by a shear stress of magnitude τ_0 applied suddenly at $y = 0$. The solution in this case is given by

$$\frac{\partial u}{\partial y} = \frac{\tau_0}{\rho \nu}(1 + \mathrm{erf}(\eta)) \quad \text{and}$$

$$u(y,t) = \frac{2\tau_0}{\rho\sqrt{\nu}} t^{1/2} \left[\eta(1 + \mathrm{erf}(\eta)) + \frac{1}{\sqrt{\pi}} \exp\left(-\eta^2\right) \right], \tag{3}$$

where ρ is the fluid density and where η is defined as above.

Aside from its value as an exact solution to the Navier–Stokes equations, Eq. (2) provides a useful paradigm for certain aspects of boundary-layer flow, and is often used pedagogically to illustrate the concepts of momentum and vorticity diffusion. The Rayleigh problem is thus a valuable starting point when we consider how non-Newtonian effects may modify the structure of unsteady boundary layers, and insight from this problem may complement that gained from studies of steady, spatially-developing boundary layers in non-Newtonian fluids [e.g. 4,5]. It is important to note that the Rayleigh problem considers the particular case in which the 'outer' flow, far from the boundary, is zero, and that in more complicated scenarios matching the inner and outer flows for non-Newtonian fluids may be a non-trivial task [5].

A large number of variations on the Rayleigh problem have been investigated, and we will not attempt to provide a comprehensive review. We will concern ourselves here only with the generalisations from Newtonian to non-Newtonian fluids, and with those studies that have sought to develop exact or asymptotic solutions rather than fully numerical solutions.

In the earliest such study, Bird [6] demonstrated that for a power-law (Ostwald–de Waele) fluid, the equation corresponding to the velocity diffusion equation (1) is non-linear, but still admits self-similar solutions analogous to Eq. (2). Bird presented solutions for several shear-thinning cases, demonstrating that the more strongly shear-thinning the fluid is, the more gradually the velocity

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decays with distance from the boundary. Wein and Mitschka [7] subsequently used this work to obtain approximate solutions to the Rayleigh problem for more general rheological models. Pascal [8,9] re-derived the power-law solution, added a yield stress, and considered the behaviour of the solutions for shear-thickening fluids. For a shear-thickening power-law fluid the boundary layer is strictly finite; that is, at any instant the velocity is identically zero beyond a certain distance from the boundary, which emerges as part of the solution to the problem [8]. An analogous finite-width boundary layer occurs at leading order in the solutions for a steady, spatially-developing boundary-layer flow in shear-thickening power-law fluids [4,5].

The Rayleigh problem has also been extensively investigated for classes of viscoelastic fluids for which the momentum equation for unsteady rectilinear flow reduces to a linear PDE. The first study of this kind was by Tanner [10], who investigated the Rayleigh problem for an Oldroyd-B fluid. Among the extensive literature that has since developed, key studies have been those by Rajagopal [11] for a second-grade fluid and by Phan-Thien and Chew [12] for a Phan-Thien–Tanner fluid. Christov [13] provides a discussion and critique of much of the more recent work on the viscoelastic Rayleigh problem. In addition, the Rayleigh problem has been extended to more complex fluids, including a model of a concentrated suspension [14] and of a nematic liquid crystal [15]; in the former case, self-similar solutions are again available.

From a physical point of view, the classical Rayleigh problem may not be the most natural one to specify, because in practice it is often easier to apply a controlled shear stress than a controlled velocity to the boundary of a fluid. Nevertheless, studies of the Rayleigh problem have confined themselves almost exclusively to the velocity-driven, rather than the shear-driven, version, perhaps because these problems are essentially equivalent in the Newtonian case. In the non-Newtonian case, however, these problems are no longer equivalent. In the present work, we will demonstrate that self-similar solutions to the shear-driven problem may be obtained for any generalised Newtonian rheology; in contrast, such solutions do not in general exist for the velocity-driven problem. Although we will illustrate the solution approach for Carreau and power-law fluids, we emphasise that it is equally applicable to *any* generalised Newtonian fluid. We will derive the form of the self-similar solutions in Section 2, and demonstrate in Section 3 how they may be used to explore some generic questions that may be asked concerning unsteady non-Newtonian boundary layers.

2. Problem specification and governing equations

2.1. Two-dimensional unsteady rectilinear flow of a generalised Newtonian fluid

The mass-conservation and momentum-balance equations for a fluid of constant density ρ, when body forces are neglected, are

$$\nabla \cdot \mathbf{u} = 0 \tag{4}$$

and

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \nabla \cdot \boldsymbol{\sigma}, \tag{5}$$

where $\mathbf{u}, p$ and $\boldsymbol{\sigma}$ are the velocity, pressure and extra-stress tensor of the fluid, and t denotes time.

A generalised Newtonian fluid is one for which the constitutive equation takes the form

$$\boldsymbol{\sigma} = 2\mu(q)\mathbf{e}, \tag{6}$$

where $\mathbf{e}$ is the rate-of-strain tensor, given by

$$\mathbf{e} = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}\right), \tag{7}$$

q is the local shear rate, given by $q = \left(2\mathrm{tr}(\mathbf{e}^2)\right)^{1/2}$, and $\mu = \mu(q)$ is a prescribed shear-rate-dependent viscosity function. The quantity $\tau = \left(\frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}^2)\right)^{1/2} = \mu(q)q$ provides a measure of the local extra stress.

For two-dimensional unsteady rectilinear flow with velocity of the form $\mathbf{u} = u(y,t)\mathbf{i}$ referred to Cartesian coordinates $Oxyz$ we have $\nabla \cdot \mathbf{u} = 0$ and $\mathbf{u} \cdot \nabla\mathbf{u} = \mathbf{0}$ identically, and the only non-zero components of $\boldsymbol{\sigma}$ are

$$\sigma_{12} = \sigma_{21} = \mu(q)\frac{\partial u}{\partial y}, \quad \text{where} \quad q = \left|\frac{\partial u}{\partial y}\right|. \tag{8}$$

Note that in such rectilinear flows the vorticity $\boldsymbol{\omega} = \omega\mathbf{k}$, where $\omega = -\partial u/\partial y$, so $q = |\boldsymbol{\omega}|$. For flows in which the pressure gradient $\partial p/\partial x$ is zero, Eq. (5) reduces to the nonlinear parabolic equation

$$\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial y}\left(\mu(q)\frac{\partial u}{\partial y}\right). \tag{9}$$

When the fluid is Newtonian, $\mu(q) = \rho\nu$, a constant, and (9) reduces immediately to (1).

2.2. The shear-driven Rayleigh problem

We consider the situation in which fluid occupies the half space $y \leqslant 0$, with a boundary at $y = 0$. Specifically, we consider the problem in which the fluid is stationary for $t < 0$ but for $t \geqslant 0$ is caused to flow with velocity $\mathbf{u} = u(y,t)\mathbf{i}$ by a constant shear stress $\tau_0 > 0$ in the x direction applied at the boundary $y = 0$. Thus at $y = 0$ we have the boundary condition

$$\sigma_{12} = \sigma_{21} = \begin{cases} 0 & \text{if} \quad t < 0, \\ \tau_0 & \text{if} \quad t \geqslant 0, \end{cases} \quad \text{or, equivalently,}$$
$$\frac{\partial u}{\partial y} = \begin{cases} 0 & \text{if} \quad t < 0, \\ q_0 & \text{if} \quad t \geqslant 0, \end{cases} \tag{10}$$

where the positive constants τ_0 and q_0 are related by

$$\tau_0 = \mu(q_0)q_0. \tag{11}$$

We seek solutions for which $\partial u/\partial y \geqslant 0$ everywhere and so we may take $q = \partial u/\partial y \geqslant 0$; the velocity u must then be maximum at $y = 0$, and we may reasonably expect the shear rate q also to be highest at the boundary, although we do not require this.

2.2.1. Nondimensionalisation

We nondimensionalise variables via

$$y = Ly^*, \quad t = \frac{\rho L^2}{\mu_{\mathrm{r}}}t^*, \quad u = \frac{L\tau_0}{\mu_{\mathrm{r}}}u^*, \quad q = \frac{\tau_0}{\mu_{\mathrm{r}}}q^*,$$
$$\tau = \tau_0\tau^*, \quad \mu = \mu_{\mathrm{r}}\mu^*, \tag{12}$$

where μ_{r} is an appropriate 'reference' viscosity (for example, the zero-shear-rate viscosity) and L is an arbitrary lengthscale. Note that the dimensionless shear stress τ^* is not an additional variable but is given by $\tau^*(q^*) = \mu^*(q^*)q^*$.

The non-dimensionalisation (12) contains the artificial lengthscale L, which remains undetermined by the boundary and initial conditions. This is an indication that self-similar solutions can be found. The only combination of y^* and t^* that is independent of L is $y^*/t^{*1/2} = (\rho/\mu_{\mathrm{r}})^{1/2}y/t^{1/2}$, while the only combination of u^* and t^* that is independent of L is $u^*/t^{*1/2} = ((\rho\mu_{\mathrm{r}})^{1/2}/\tau_0)u/t^{1/2}$. Requiring that when a solution $u(y,t)$ is non-dimensionalised the corresponding solution $u^*(y^*,t^*)$ should not depend on L thus leads us to consider solutions of the self-similar form

$$u^* = 2t^{*1/2}f(\eta), \quad \text{where} \quad \eta = \frac{y^*}{2t^{*1/2}}, \tag{13}$$

and where the factors of 2 have been introduced for convenience. For solutions of this form, $q^* = f'(\eta)$ and $\mu^* = \mu^*(f')$ are also independent of L.