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The polar analysis of the Third-order Shear Deformation Theory of laminates

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ABSTRACT

In this paper the Verchery's polar method is extended to the conceptual framework of the Reddy's Third-order Shear Deformation Theory (TSDT) of laminates. In particular, a mathematical representation based upon tensor invariants is derived for all the laminate stiffness matrices (basic and higher-order stiffness terms). The major analytical results of the application of the polar formalism to the TSDT of laminates are the generalisation of the concept of a *quasi-homogeneous* laminate as well as the definition of some new classes of laminates. Moreover, it is proved that the elastic symmetries of the laminate shear stiffness matrices (basic and higher-order terms) depend upon those of their in-plane counterparts. As a consequence of these results a unified formulation for the problem of designing the laminate elastic symmetries in the context of the TSDT is proposed. The optimum solutions are found within the framework of the polar-genetic approach, since the objective function is written in terms of the laminate polar parameters, while a genetic algorithm is used as a numerical tool for the solution search. In order to support the theoretical results, and also to prove the effectiveness of the proposed approach, some new and meaningful numerical examples are discussed in the paper.

1. Introduction

As well known, the Classic Laminate Theory (CLT) together with the First-order Shear Deformation Theory (FSDT) are the simplest theories employed for describing the mechanical behaviour of a composite laminate considered as an equivalent homogeneous (generally) anisotropic plate. Such theories properly describe the laminate kinematic response in the case of small (CLT) or moderate (FSDT) values of the plate characteristic aspect ratio (i.e. the ratio of its thickness to its shorter side). However, the major drawback of these theories is in the estimation of the influence of the laminate transverse shear stiffness on its mechanical response (which becomes more and more important for thick plates). On one hand, in the case of the CLT the laminate transverse shear stiffness does not intervene in the definition of the laminate constitutive equation (making this theory adequate only for thin laminates). On the other hand, in the framework of the FSDT the influence of the transverse shear stiffness is taken into account within the definition of the laminate constitutive behaviour. Nevertheless, due to the kinematic model on which the FSDT relies, the through-the-thickness shear stresses are constant within each constitutive layer, leading in this way to a mechanical contradiction. Indeed, the shear stresses do not satisfy: (a) the boundary conditions on the external faces of the laminate, (b) the local equilibrium equations (elasticity solution) and (c) the continuity condition at the layers interface, see [1]. To overcome these contradictions, it is common to introduce the so-called "shear correction factor" [1,2] which generally satisfies only two of the previous three conditions. However, in the context of the FSDT, the definition of the shear correction factor is immediate only for isotropic plates, while it becomes more arduous defining such a quantity for a laminate since it depends upon the geometrical parameters of the stack (layers orientations and positions) [1].

Higher order theories allow for overcoming such a difficulty: they give a better description of both the laminate kinematics and stress field without the need of introducing any correction coefficient. However these theories require the introduction of higher-order stress resultants and stiffness matrices whose physical meaning is not immediate. In literature one can find several higher-order theories of different nature: for each theory the displacement field is expanded in a finite series (in terms of the thickness coordinate) of unknown functions: the terms of the series (i.e. the functions depending upon the thickness coordinate) can belong to a given basis (polynomial, trigonometric, radial, B-spline,

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Notations

CLT classical laminate theory
FSDT First-order Shear Deformation Theory
TSDT Third-order Shear Deformation Theory
GA genetic algorithm
$\Gamma = \{O; x_1, x_2, x_3\}$ local (or material) frame of the elementary ply
$\Gamma^l = \{O; x, y, z = x_3\}$ global frame of the laminate
θ rotation angle
$\{11, 22, 33, 32, 31, 21\} \Longleftrightarrow \{1, 2, 3, 4, 5, 6\}$ correspondence between tensor and Voigt's (matrix) notation for the indexes of tensors (local frame)
$\{xx, yy, zz, zy, zx, yx\} \Longleftrightarrow \{x, y, z, q, r, s\}$ correspondence between tensor and Voigt's (matrix) notation for the indexes of tensors (global frame)
Z_{ij}, $(i, j = 1, 2$ or $i, j = x, y)$ second-rank plane tensor using tensor notation (local and global frame)
L_{ijkl}, $(i, j, k, l = 1, 2$ or $i, j, k, l = x, y)$ fourth-rank plane tensor using tensor notation (local and global frame)
u, v, w components of the laminate displacement field within the global frame Γ^l
$u_0, v_0, w_0, \phi_x, \phi_y$ the five independent kinematic unknowns in the context of the Reddy's TSDT
n number of layers
$\{\delta_k\}$ $(k = 1, \ldots, n)$ vector of the layers orientation angles
z_{k-1}, z_k thickness coordinates of bottom and top faces of the kth constitutive ply, respectively
h overall thickness of the laminate
$\{\varepsilon^{(0)}\}, \{\varepsilon^{(1)}\}, \{\varepsilon^{(3)}\}, 3 \times 1$ vectors of in-plane strains of the laminate middle plane
$\{\gamma^{(0)}\}, \{\gamma^{(2)}\}, 2 \times 1$ vectors of the transverse shear strains of the laminate middle plane
$\{\mathrm{N}\}, \{\mathrm{M}\}, \{\mathrm{P}\}, 3 \times 1$ vectors of higher-order generalised in-plane forces (per unit length)
$\{\mathrm{Q}\}, \{\mathrm{R}\}, 2 \times 1$ vectors of higher-order generalised transverse shear forces (per unit length)
$[\mathrm{Q}], 3 \times 3$ in-plane reduced stiffness matrix of the constitutive lamina
$[\widehat{\mathrm{Q}}], 2 \times 2$ out-of-plane reduced stiffness matrix of the constitutive lamina
$T_0, T_1, R_0, R_1, \Phi_0, \Phi_1$ polar parameters of a fourth-rank plane tensor (also used for the lamina in-plane reduced stiffness matrix $[\mathrm{Q}]$)
T, R, Φ polar parameters of a second-rank plane tensor (also used for the lamina transverse shear reduced stiffness matrix $[\widehat{\mathrm{Q}}]$)
$[\mathrm{A}], [\mathrm{B}], [\mathrm{D}], [\mathrm{E}], [\mathrm{F}], [\mathrm{H}], 3 \times 3$ in-plane stiffness matrices of the laminate (membrane, membrane/bending coupling, bending and higher-order stiffness, respectively)
$[\mathrm{A}^*], [\mathrm{B}^*], [\mathrm{D}^*], [\mathrm{E}^*], [\mathrm{F}^*], [\mathrm{H}^*], 3 \times 3$ homogenised in-plane stiffness matrices of the laminate (membrane, membrane/bending coupling, bending and higher-order stiffness, respectively)
$[\widehat{\mathrm{A}}], [\widehat{\mathrm{D}}], [\widehat{\mathrm{F}}], 2 \times 2$ transverse shear stiffness matrices of the laminate (basic and higher-order stiffness, respectively)
$[\widehat{\mathrm{A}}^*], [\widehat{\mathrm{D}}^*], [\widehat{\mathrm{F}}^*], 2 \times 2$ homogenised transverse shear stiffness matrices of the laminate (basic and higher-order stiffness, respectively)
b_k, d_k, e_k, f_k, h_k coefficients of the laminate stiffness matrices
$T_{0M^*}, T_{1M^*}, R_{0M^*}, R_{1M^*}, \Phi_{0M^*}, \Phi_{1M^*}$ polar parameters of the generic homogenised in-plane stiffness matrix of the laminate $(M^* = A^*, B^*, D^*, E^*, F^*, H^*)$
$T_{\widehat{M}^*}, R_{\widehat{M}^*}, \Phi_{\widehat{M}^*}$ polar parameters of the generic homogenised transverse shear stiffness matrix of the laminate $(\widehat{M}^* = \widehat{A}^*, \widehat{D}^*, \widehat{F}^*)$
$[\mathrm{C}_1^*], [\mathrm{C}_2^*], [\mathrm{C}_3^*], 3 \times 3$ laminate homogeneity matrices
E_i, $(i = 1, 2, 3)$ Young's moduli of the constitutive lamina (material frame)
G_{ij}, $(i, j = 1, 2, 3)$ shear moduli of the constitutive lamina (material frame)
ν_{ij}, $(i, j = 1, 2, 3)$ Poisson's ratios of the constitutive lamina (material frame)
t_{ply} thickness of the constitutive lamina
Ψ overall objective function for the problem of designing the elastic symmetries of the laminate
$\{\mathrm{f}\}, 37 \times 1$ vector of partial objective functions
$[\mathrm{W}], 37 \times 37$ positive semi-definite diagonal weight matrix
$\widehat{R_{0M^*}}, \widehat{R_{1M^*}}, \widehat{\Phi_{0M^*}}, \widehat{\Phi_{1M^*}}$ imposed values for the polar parameters of matrix $[\mathrm{M}^*], (\mathrm{M}^* = \mathrm{A}^*, \mathrm{D}^*, \mathrm{F}^*, \mathrm{H}^*)$
N_{pop} number of populations
N_{ind} number of individuals
N_{gen} number of generations
p_{cross} crossover probability
p_{mut} mutation probability

NURBS, etc.). In principle it is possible to expand the displacement field up to any degree in terms of the thickness coordinate. Nevertheless, an expansion up to the third order (the so-called third-order theory) is sufficient to capture the quadratic variation of the transverse shear strains and stresses within each layer. There are a lot of papers on third-order theories, see for instance [3–11]. Despite they seem to differ from each other, the displacement fields of these theories are mechanically equivalent (or related), see [12]. Recently, the classical Third-order Shear Deformation Theory (TSDT) of laminates, initially introduced by Reddy [3], has been extended and reformulated according to the Eringen's nonlocal linear elasticity theory to capture small scale size effects through the thickness [13].

The aim of this paper does not consist in a critical analysis of all the different types of TSDT that can be found in literature, rather it aims to shed some light on certain aspects linked to the formulation of the laminate constitutive equation in the conceptual framework of the classical TSDT of Reddy [1]. Particularly, the objective of the present work is twofold: on one hand it aims of clarifying the physical meaning of the higher-order stiffness matrices while on the other hand it intends of estimating their influence on the elastic response of the laminate. To these purposes the polar method initially introduced by Verchery [14], later enriched and deeply investigated by Vannucci and his co-workers [15–19] and recently extended to the FSDT of laminates [20] is here employed (for the first time) within the framework of the TSDT. In particular, the expression of the polar parameters of the laminate higher-order stiffness matrices is analytically derived. Thanks to the polar formalism and its application to the TSDT it is possible to introduce some new classes of laminates and also to generalise the definition of a *quasi-homogeneous* laminate, initially introduced by Vannucci and Verchery [21]. Accordingly, it is possible to carry out a more general analysis of the elastic response of the laminate by reformulating and generalising the problem of designing its elastic symmetries (initially introduced by Vannucci [22] and later extended to the FSDT [20]) within the context of the TSDT. This problem is formulated as an unconstrained minimisation problem in the space of the full set of the laminate polar parameters (even including the higher-order stiffness matrices). Due to its particular nature (i.e. a non-convex optimisation problem in the space of the layers orientation angles), the solution search process is performed by using the genetic algorithm (GA) BIANCA [23–25]. Finally, in