Journal of Non-Newtonian Fluid Mechanics 187–188 (2012) 32–35

Contents lists available at SciVerse ScienceDirect

Journal of Non-Newtonian Fluid Mechanics

journal homepage: http://www.elsevier.com/locate/jnnfm

Short Communication

# Fractional time-dependent Bingham model for muddy clay

Deshun Yin *, Wei Zhang, Chen Cheng, Yanqing Li

*College of Mechanics and Materials, Hohai University, No. 1, Xikang Road, Nanjing, Jiangsu 210098, PR China*

ARTICLE INFO

*Article history:*
Received 5 July 2012
Received in revised form 6 September 2012
Accepted 6 September 2012
Available online 25 September 2012

*Keywords:*
Fractional derivative
Time dependent
Bingham model
Muddy clay

ABSTRACT

In this short communication, based on fractional calculus, we present the fractional Bingham model which can describe the time dependent behavior in the fluid with yield strength. The pulling sphere tests under three different speed conditions are be used to investigate the time dependence of muddy clay. The experimental results illustrate the muddy clay is a noticeable time-dependent Bingham fluid. Experimental results show that the fractional Bingham model can well depict the mechanical behaviors of the muddy clay, and give a good agreement with the experimental data. In addition, this study also finds that the order of the fractional Bingham model is larger than 1 in some cases, which breaks the commonly used assumption that the order should be in the interval [0,1].



## 1. Introduction

With the increase of water volume fraction, the state of soil changes from hard to soft. When soil is solid, it can display evidently the time dependence, such as the alteration of rate in loading can have a manifest effect on the experimental results [1]. Thus, a large number of viscoelastic, viscoplastic or elasto-viscoplastic constitutive models were developed to describe the time dependent property, of which Maxwell model, Kelvin–Voigt model and standard linear solid model are the most commonly used ones [2]. On the other hand, with the increase of moisture content, soil turn into flow matter such as the artificial soil suspensions and the natural debris flow, appearing clear non-Newtonian fluid characters. Through the test observation, it is known that mud [3–5], debris flow [6,7], liquefied soil [8] and clay suspensions [9,10] exhibit the Bingham plastic behavior, which is characterized by a yield stress. Therefore, Bingham model is the most common one for such non-Newtonian fluids [4,8,11–17]. Some researchers also gave several modified Bingham models due to some special requirements, for example, Garrido et al. [18] employed a double ternary Bingham model to give the quantitative and satisfactory explanation of this complex rheological behavior, and a power law and a Bingham model is implemented for the flow of bulk materials by Leonardi et al. [19]. Because the Bingham model mentioned above is time independent and only focus on the impact of the magnitude of strain rate, it is neither concerned with the effect of the loading time length, nor with the influence of the strain rate change with time. However, experimental observations show that some muddy clay and clay dispersions with yield stress can display evidently time dependent rheological behavior [9]. Thus, it is necessary to develop a new model to describe time-dependence in Bingham plastic fluid.

Fractional calculus allows one to define precisely non-integer order integrals or derivatives. It has been found that fractional calculus is a powerful tool for modeling the viscoelastic behaviors and non-Newtonian fluid characteristics and particularly suited for building the time-dependent constitutive model. The main reason for this development is that a fractional model could describe simply and elegantly the complex characteristics of a viscoelastic material. Therefore, the classical integer order Maxwell model, Kelvin–Voigt model and standard linear solid model have been upgraded to the fractional order ones for viscoelastic materials [20]. The rheological constitutive equations with fractional derivatives play an important role in the description of the behavior of the polymer solutions and melts. The starting point of the fractional derivative model of a non-Newtonian fluid is also a classical differential equation which is modified by replacing the time derivative of an integer order by the fractional calculus operators. Shan et al. [21] used the fractional order derivative to establish the relaxation models for non-Newtonian viscoelastic fluids in dual porous media to research the seepage flow. Ezzat [22] constructed a new mathematical model for thermoelectric MHD non-Newtonian fluid with fractional order. Mahmood et al. [23] gave the exact analytic solutions for the unsteady flow of a non-Newtonian fluid between two cylinders by a fractional derivative model. In other cases, it has been shown that the constitutive equations employing fractional derivatives are also linked to molecular theories [24]. However, until now, it has not been found that the Bingham model is modified using fractional derivative.

* Corresponding author. Tel.: +86 13675176567.
*E-mail address:* yindeshun@hhu.edu.cn (D. Yin).

0377-0257/$ - see front matter © 2012 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.jnnfm.2012.09.003

The aim of this study is to improve upon the current Bingham model by fractional calculus, allowing the new model to describe time dependence. In this paper, the pulling sphere experiment of muddy clay was carried out under the condition of constant strain rate and variable strain rate with time. The mechanical behaviors of muddy clay are analyzed by the proposed fractional model.

## 2. Fractional Bingham model

It is well known that ideal solids obeys Hooke's law, $\tau(t)$–$\varepsilon(t)$, and Newtonian fluids satisfy Newton's law of viscosity, $\tau(t)$–$d^1\varepsilon(t)/d^1t$, where $\tau$ is stress, $\varepsilon$ is strain and $t$ represents time. $d^1\varepsilon(t)/d^1t$ is the first derivative of $\varepsilon(t)$ to time, i.e. the strain rate $\dot{\varepsilon}(t)$. So it is not difficult to imagine that an "intermediate" material, which is intermediate between ideal solid and Newtonian fluid, should follow

$$\tau(t) = \theta\lambda^\alpha \frac{d^\alpha \varepsilon(t)}{dt^\alpha}, \ (0 \leqslant \alpha \leqslant 1) \tag{1}$$

here, $\lambda$ and $\theta$ are material constants. Eq. (1) is often employed to construct the component models, such as fractional Maxwell model, and fractional Kelvin–Voigt model. Because there exists the time fractional derivative in Eq. (1), the fractional model has a strong time dependence.

The ideal Bingham model and the Herschel–Bulkley model are the most common for fluid soil. The ideal Bingham model is written as

$$\tau = \tau_y + \mu\dot{\varepsilon} \tag{2}$$

where $\tau_y$, $\mu$, and $\dot{\varepsilon}$ represent yield strength, dynamic viscosity, and shear strain rate, respectively. The model requires that motion of the fluid does not begin until the yield strength of the fluid is exceeded, after which the fluid flows as a Newtonian fluid with a linear stress–strain rate relationship determined by the dynamic viscosity of the fluid.

The Herschel–Bulkley model is as follows:

$$\tau = \tau_y + \eta\dot{\varepsilon}^n \tag{3}$$

where $\eta$ represent 'viscosity' and the exponent $n$ is adjusted to fit measured data. When the exponent $n \to 1$, $\eta \to \mu$. Clearly, the Herschel–Bulkley model can depict the nonlinear relationship between stress and strain rate, which can represent that the fluid display the non-Newtonian fluid property, when stress is over the yield stress.

In order to enable Bingham model to involve the time affecting factor, through putting Eq. (1) into Eq. (2), we define the fractional Bingham model as

$$\tau(t) = \tau_y + \theta\lambda^\alpha \frac{d^\alpha \varepsilon}{dt^\alpha} \tag{4}$$

here, $\varepsilon$ is shear strain.

Since the main concern in fluid mechanics is the relationship between stress and strain rate but not the stress–strain one, Eq. (4) can be rewritten as

$$\tau(t) = \tau_y + \theta\lambda^\alpha \frac{d^{\alpha-1}\dot{\varepsilon}}{dt^{\alpha-1}} \tag{5}$$

In this paper, we pay main attention to three different variation condition of strain rate with time, $\dot{\varepsilon} = c$, $\dot{\varepsilon} = ct$, and $\dot{\varepsilon} = ct^2$, where $c$ represents constant.

When $\dot{\varepsilon} = c$, based on Riemann–Liouville fractional calculus, Eq. (5) can be expressed as

$$\tau(t) = \tau_y + \theta\lambda^\alpha \frac{ct^{1-\alpha}}{\Gamma(2-\alpha)} \tag{6}$$

here, $\Gamma(*)$ is gamma function.

For a long time, it is always thought that $\alpha$ vary between 0 and 1. Hence, we here also provisionally assume that $\alpha$ is greater than 0 and less than 1.

If $0 < \alpha < 1$, it is known from Eq. (6) that the stress is not constant but is gradually increasing with time when strain rate remain unchanged. This indicates that the fractional Bingham model can describe the time dependence.

When strain rate is linear increasing with time, $\dot{\varepsilon} = ct$, the fractional Bingham model can be

$$\tau(t) = \tau_y + \theta\lambda^\alpha \frac{ct^{2-\alpha}}{\Gamma(3-\alpha)} \tag{7}$$

The strain rate form of Eq. (7) is

$$\tau(\dot{\varepsilon}) = \tau_y + \theta\lambda^\alpha \frac{c^{\alpha-1}\dot{\varepsilon}^{2-\alpha}}{\Gamma(3-\alpha)} \tag{8}$$

Clearly, when $0 < \alpha < 1$, stress will quickly increase with the rising strain rate due to exponent $2-\alpha > 1$.

For $\dot{\varepsilon} = ct^2$, Eq. (5) can be transformed into

$$\tau(t) = \tau_y + \theta\lambda^\alpha \frac{ct^{3-\alpha}}{\Gamma(4-\alpha)} \tag{9}$$

and

$$\tau(\dot{\varepsilon}) = \tau_y + \theta\lambda^\alpha \frac{c^{(\alpha-1)}/2\dot{\varepsilon}^{(3-\alpha)}/2}{\Gamma(3-\alpha)} \tag{10}$$

Analogously with the case of $\dot{\varepsilon} = ct$, when $0 < \alpha < 1$ and $\dot{\varepsilon} = ct^2$, stress will also rapidly grow with the evolution of time.

Clearly, under the assumption of $0 < \alpha < 1$, when $\dot{\varepsilon} = ct$ or $\dot{\varepsilon} = ct^2$, in accordance with the fractional model, a shear thicken behavior will be following after the applied stress is over the yield strength. If we suppose that $\alpha$ can be greater than 1 and less than two, the fluid will flow as a shear thinning fluid because the corresponding exponent of strain rate in Eq. (8) and (10) is less than 1.

Perhaps one regards the Herschel–Bulkley Bingham model and the fractional model as the same since Eqs. (3), (8) and (10) all have the exponent shape of strain rate; nevertheless, the exponent form just as in Eqs. (8) and (10) can obtained from the fractional Bingham model only when the strain rate is the special time function, such as $\dot{\varepsilon} = ct^m$, where $m$ is a real number. In other words, the relationship between time and strain rate determines the stress functional form, derived from the fractional Bingham model, while the Herschel–Bulkley Bingham model do not take into account the relationship between time and strain rate. This clearly shows that the fractional Bingham model owns time dependence.

## 3. The pulling sphere test

In our study, the pulling sphere test in muddy clay was carried out for model validation. The detail on this experiment will be introduced in this section.

We measured the force $F$ on a steel sphere while and after it moved horizontally through a reconstituted muddy clay. The experimental apparatus shown schematically in Fig. 1 is modified on the basis of that described in Ref. [25]. The reconstituted muddy clay was contained in a rectangular strengthened glass container 550 mm long, 340 mm wide and 400 mm tall, closed at the bottom but open at the top. The rectangular container is large enough that wall effects are not expected to be important. A 40 mm steel sphere was attached to a calibrated load cell by a length of monofilament nylon thread. The load cell had a maximum capacity of 20 kg and a response time that was faster than the interval between recorded data points. It was mounted on a linear actuator which was driven by a computer controlled servomotor. The