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A layerwise *p*-version finite element formulation for free vibration analysis of thick composite laminates with curvilinear fibres

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ABSTRACT

Herein, modes of vibration of a novel type of composite laminates, manufactured by tow-placement machines and which are called Variable Stiffness Composite Laminates (VSCL), are of interest. Layerwise theory is chosen, because it leads to an accurate prediction of displacements through the thickness of laminates, even when these are thick. Furthermore, it naturally has the capability of modelling the bending-membrane coupling that occurs on free vibration of VSCL plates with unsymmetric stacking sequences. The continuity of displacements at layer interfaces is imposed and in-plane displacements vary in a zig-zag fashion along the thickness. The accuracy of the model is confirmed by testing Constant Stiffness Composite Laminates (CSCL) analysed in the literature by others, who employed either Equivalent Single Layer (ESL) or layerwise theories. Abaqus commercial finite element software is as well employed in order to test the present approach. Finally, published natural frequencies of VSCLs, which were studied by an ESL theory, are compared with frequencies resulting from the present layerwise theory. The verification of the proposed model is followed by an investigation on the effect of curvilinear fibre orientation parameters on natural frequencies of thin to thick composite plates, with symmetric and unsymmetric lay-ups.

1. Introduction

The development of manufacturing technologies for composite materials has received much attention over the past few decades. Advanced fibre-reinforced composite materials can be produced by changing their macro-scale characteristics, one of which is the fibre orientation, which can be varied along a curvilinear path to produce Variable Stiffness Composite Laminates (VSCL). This idea was proposed by Hyer in [1,2], to overcome the discontinuities that occur when the fibre orientation is changed from one part of a structure to another. Later, manufacturing possibilities of tow placement machines were explored by Waldhart in [3]. Manufacturing VSCL with parallel fibre paths was proposed and the in-plane responses of shifted and parallel VSCL were compared.

Various analyses have been performed to study the structural behaviour of VSCL [4], of which we recall investigations devoted to analyse vibrations of VSCL plates. Classic Laminated Theory (CLT) was applied to study the vibration of VSCL plates in [5–7]. However, CLT is applicable only for thin laminates due to the lack of transverse shear strains. More recently, the attention has been drawn toward thick VSCL plates, in which one has to utilise shear deformation laminated theories. Non-linear vibration of VSCL plates was analysed by employing a First order Shear Deformation Theory (FSDT) in [8,9]. A Third order Shear Deformation Theory (TSDT) was employed in [10] to study the free vibration of thin to moderately thick VSCL plates. However, both FSDT and TSDT theories are Equivalent Single Layer (ESL) theories, which can fail to predict the response of considerably thick laminates [11]. More advanced theories are required to better predict the cross section warping and displacement distributions through the thickness. Herein, a layerwise theory is of interest, in which each layer is modelled as an independent plate and the continuity of displacement is imposed at the layers' interfaces [12]. Layerwise theory should lead to more accurate vibration analysis. Furthermore, with a layerwise model the analysis of laminates that are not symmetric about their middle plane is straightforward and, to the best of authors' knowledge, vibration of unsymmetric VSCL plates was only considered once, in [13], where an ESL-CLT model, applicable to thin plates, was employed.

This work aims to analyse the linear modes of vibration of thin to thick VSCL plates, based on a layerwise theory and *p*-version finite element method, which is an extension of the element used to study the non-linear deflection, in the static regime, of VSCL plates in [14]. At first, the proposed model is verified. To do so,

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Abaqus commercial software and the published literature available on layerwise or ESL higher order theories is utilised. Then, taking benefit of the capability of modelling thick composite laminates with any lay-ups, thick symmetric and unsymmetric VSCL plates are studied. The effect of fibre orientation on the modes of vibration of VSCL plates is investigated by computing the natural frequencies of vibration and plotting the mode shapes.

2. Formulation

First order displacement fields are adopted for each individual layer. The layers are considered to be perfectly bonded at the interfaces by imposing continuity of displacements. The displacement field for the layer kth is presented as

$$\begin{aligned}U^k(x,y,z_k,t)&=\frac{1}{2}(u^k(x,y,t)+u^{k+1}(x,y,t))-\frac{z_k}{h_k}(u^{k+1}(x,y,t)-u^k(x,y,t)),\\V^k(x,y,z_k,t)&=\frac{1}{2}(v^k(x,y,t)+v^{k+1}(x,y,t))-\frac{z_k}{h_k}(v^{k+1}(x,y,t)-v^k(x,y,t)),\\W^k(x,y,z_k,t)&=W^k(x,y,t),\end{aligned}\tag{1}$$

where u, v, and w with superscripts k and $k+1$ are the translations of layer kth at its bottom and top side, respectively. h_k is the thickness of layer kth and t is time. Each layer is modelled with respect to its Cartesian coordinate system.

The three displacement components of a point in each individual layer may be expressed in terms of polynomial shape functions $N_i^j(\xi,\eta), i=1-n, j=u,v,w$, in the local coordinate system and unknown coefficients $\mathbf{q}(t)$, which are determined later, by solving the equations of motion

$$\begin{aligned}u^k(\xi,\eta,t)&=\sum_{i=1}^{n}N_i^u(\xi,\eta)q_i^{u^k}(t),\\v^k(\xi,\eta,t)&=\sum_{i=1}^{n}N_i^v(\xi,\eta)q_i^{v^k}(t),\\w^k(\xi,\eta,t)&=\sum_{i=1}^{n}N_i^w(\xi,\eta)q_i^{w^k}(t).\end{aligned}\tag{2}$$

A couple of sets of polynomial shape functions is employed: one set, $N_i^u(\xi,\eta)=N_i^v(\xi,\eta)$, for the in-plane displacement components; another set, $N_i^w(\xi,\eta)$, for the transverse displacements [15,16]. The accuracy of the model depends on two distinct issues. On the one hand, it depends on the number of individual layers employed, that is, on the upper limit of k in Eq. (2) and Fig. 1. On the other hand, accuracy can be increased by using more polynomial shape functions and generalised coordinates for each layer of the model, *i.e.* by increasing n in Eq. (2). Hence, the total number of degrees of freedom in the formulation is associated both with the number of layers of the model – which is not necessarily the number of composite plies – and the number of shape functions. It is mentioned that it was here decided to use the same number of shape functions for in-plane and out-of-plane displacement components.

This type of finite element method, where to increase accuracy one increases the number and order of the shape functions employed, keeping the mesh unchanged, is known as p-version finite element method. It is noted that due to the use of high order shape functions, shear locking does not occur. In addition, unlike what happens in ESL theories, in a layerwise formulation the dimension of the global stiffness and mass matrices can be altered by increasing the number of layers of the model, and consequently, more accurate models can be derived.

Because rectangular plates are analysed here, one p-element is sufficient; hence the global and local coordinates are related by Eq. (3).

$$\begin{aligned}x&=\frac{a\xi}{2},\\y&=\frac{b\eta}{2},\end{aligned}\tag{3}$$

where a and b are the length and width of the laminate, whereas ξ and η denote a non-dimensional coordinate system.

The strain tensor can be divided into in-plane and out-of-plane portions, which in the linear regime appear as shown in the following equations

$$\begin{aligned}\varepsilon_x^k&=\frac{1}{2}\left(u_{,x}^{k+1}(x,y,t)+u_{,x}^k(x,y,t)\right)+\frac{z_k}{h_k}\left(u_{,x}^{k+1}(x,y,t)-u_{,x}^k(x,y,t)\right),\\\varepsilon_y^k&=\frac{1}{2}\left(v_{,y}^{k+1}(x,y,t)+v_{,y}^k(x,y,t)\right)+\frac{z_k}{h_k}\left(v_{,y}^{k+1}(x,y,t)-v_{,y}^k(x,y,t)\right),\\\gamma_{xy}^k&=\frac{1}{2}\left(u_{,y}^{k+1}(x,y,t)-u_{,y}^k(x,y,t)+v_{,x}^{k+1}(x,y,t)-v_{,x}^k(x,y,t)\right)\\&\quad+\frac{z_k}{h_k}\left(u_{,y}^{k+1}(x,y,t)-u_{,y}^k(x,y,t)+v_{,x}^{k+1}(x,y,t)-v_{,x}^k(x,y,t)\right),\\\gamma_{xz}^k&=w_{,x}^k+\frac{1}{h_k}(u^{k+1}(x,y,t)-u^k(x,y,t)),\\\gamma_{yz}^k&=w_{,y}^k+\frac{1}{h_k}(v^{k+1}(x,y,t)-v^k(x,y,t)),\end{aligned}\tag{4}$$

where a comma denotes a partial derivative with respect to the coordinate indicated.

The stress–strain relation for kth composite layer follows Hooke's law in the local coordinate system is presented as [17]

$$\begin{Bmatrix}\sigma_x\\\sigma_y\\\tau_{xy}\\\tau_{yz}\\\tau_{xz}\end{Bmatrix}^{(k)}=\begin{bmatrix}\overline{Q}_{11}&\overline{Q}_{12}&\overline{Q}_{16}&0&0\\\overline{Q}_{12}&\overline{Q}_{22}&\overline{Q}_{26}&0&0\\\overline{Q}_{16}&\overline{Q}_{26}&\overline{Q}_{66}&0&0\\0&0&0&\overline{Q}_{44}&\overline{Q}_{45}\\0&0&0&\overline{Q}_{45}&\overline{Q}_{55}\end{bmatrix}^{(k)}\begin{Bmatrix}\varepsilon_x\\\varepsilon_y\\\gamma_{xy}\\\gamma_{yz}\\\gamma_{xz}\end{Bmatrix}^{(k)}=\overline{\mathbf{Q}}\boldsymbol{\varepsilon}.\tag{5}$$

In Eq. (5), $\overline{\mathbf{Q}}$ is the reduced stiffness matrix in a lamina and results from the transformation of matrices from the nominal direction to the local coordinate system. The transformation is associated to the fibre orientation within the plane of the laminate. Although function arguments were not included in Eq. (5) – in order to simplify the notation, the same simplification is adopted in other equations – it is important to point out that the terms of the reduced stiffness matrix are not constant, they are, in general, functions of Cartesian coordinates x and y. Here, a linear variation of the fibre angle is adopted, as in [18,19] and as shown in Fig. 1. Hence, the reference path can be defined by parameters T_0, the fibre orientation angle at the origin, and T_1, which denotes the fibre orientation angle at the edge of each ply

$$\theta(x)=\frac{2(T_1-T_0)}{a}\mid x\mid+T_0.\tag{6}$$

Therefore, the reduced stiffness matrix, which relates stresses and strains [14], is a function of x, in this paper.

The equation of motion is obtained by applying the principle of virtual work

$$\delta W_{in}+\delta W_{ext}+\delta W_v=0,\tag{7}$$

where δW_{in}, δW_{ext}, and δW_V are the virtual work of internal, external and inertia forces and are defined as

$$\begin{aligned}\delta W_{in}&=-\int_V\delta\boldsymbol{\varepsilon}^T\boldsymbol{\sigma}dV\\&=-\int_{\Omega_0}\left\{\int_{-\frac{h}{2}}^{\frac{h}{2}}\left[\sigma_x\delta\varepsilon_x+\sigma_y\delta\varepsilon_y+\tau_{xy}\delta\gamma_{xy}+\tau_{xz}\delta\gamma_{xz}+\tau_{yz}\delta\gamma_{yz}\right]dz\right\}d\Omega_0\\&=-\delta\mathbf{q}^T\mathbf{K}(\mathbf{q})\mathbf{q},\\\delta W_V&=-\int_{\Omega_0}\left\{\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho_0(\ddot{u}\delta u+\ddot{v}\delta v+\ddot{w}\delta w)dz\right\}d\Omega_0=-\delta\mathbf{q}^T\mathbf{M}\ddot{\mathbf{q}},\end{aligned}\tag{8}$$