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A note on the pipe flow with a pressure-dependent viscosity

Eduard Marušić-Paloka, Igor Pažanin *

Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička 30, 10000 Zagreb, Croatia

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ABSTRACT

We study a stationary motion of incompressible viscous fluid with a pressure-dependent viscosity in a thin (or long) straight pipe with variable cross-section. We assume that the flow is governed by the prescribed pressure drop between pipe's ends. Under very general assumption on the viscosity law satisfied by the Barus formula and other empiric laws, the effective behavior of the flow is found via rigorous asymptotic analysis with respect to the pipe's thickness.

1. Introduction

The idea of fluid with pressure-dependent viscosity goes back to Stokes and one of his celebrated work [1] published in 1845. Since then, numerous researchers measured the effects of pressure-dependent viscosity for various liquids and regimes of the flow (see e.g. [2–7]). It turns out that, especially at high values of pressure, the dependence of viscosity on pressure becomes important while the flow is still incompressible. Therefore, it makes sense to investigate the incompressible flow of fluids with a pressure-dependent viscosity.

Several models have been used to describe the viscosity-pressure relation. The most common throughout the literature are the linear law

$$\mu(p) = \alpha p, \quad \text{or } \mu(p) = \mu_0(1 + \alpha p) \tag{1}$$

and the exponential law

$$\mu(p) = \mu_0 e^{\alpha p} \tag{2}$$

known as the Barus formula. Here μ_0 represents viscosity at atmospheric pressure while $\alpha > 0$ is the pressure-viscosity coefficient. As reported by Denn [8], the coefficient α is, approximately, of order $3 \times 10^{-8}\ \mathrm{Pa}^{-1}$ for polymer melts and viscosity law (2). For mineral oils, α typically ranges between 10^{-8} and 2×10^{-8} (see Jones [9]). Barus formula (2) has been extensively used by engineers, sometimes combined with temperature dependence. Some other possibilities for pressure (and shear) dependent viscosity relations can be found in papers [10,11] by Rajagopal and his collaborators.

In case of linear law (1), the explicit exact solutions of the equations of motion have been reported for some simplified situations like unidirectional and plane-parallel flows, see Renardy [12], Hron et al. [13], Vasudevaiah and Rajagopal [14] and Kalogirou et al. [15]. The exponential law (2) rules out the possibility of deriving the exact solutions (even for simple flows) and that is the main reason it has been avoided in theoretical analyses presented in the literature. The another reason lies in the fact that not much has been done in proving the well-posedness of the corresponding boundary-value problems in case of viscosity law (2). Indeed, the existence results for incompressible fluid flows with pressure-dependent viscosity have been provided only under certain technical assumptions on the viscosity which are not fulfilled by the Barus formula (2). We refer the reader to Renardy [16], Gazzola [17] and Malek et al. [10,11]. However, very recently, Marušić-Paloka in [18] managed to prove the well-posedness of the stationary Stokes system under very general assumption on $\mu(p)$ satisfied by (2) and other empiric laws. More precisely, assuming that the growth of the function $p \mapsto \mu(p)$ and its derivative is at most exponential, i.e.

$$0 < \mu(p) \leqslant C_1\, e^{\alpha p}, \quad 0 \leqslant \mu'(p) \leqslant C_2\, e^{\alpha p}, \quad (C_1, C_2 = \text{const.} > 0), \tag{3}$$

it is shown that the corresponding Dirichlet problem has a solution, which is unique under some technical condition[1] which does not rule out the Barus formula.

[1] It is well known that in classical Stokes (or Navier–Stokes) system the pressure is determined only up to constant. To assure uniqueness we need to impose some additional condition, like prescribing the value of its integral over a specified domain. According to [18], it turns out that similar, slightly more technical condition is also needed here to fix the pressure.

* Corresponding author. Tel.: +385 14605866.
E-mail addresses: emarusic@math.hr (E. Marušić-Paloka), pazanin@math.hr (I. Pažanin).

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The goal of this paper is to study incompressible fluid flow through a straight pipe with variable cross-section and with viscosity $\mu(p)$ obeying (3). The flow is assumed to be governed by the prescribed pressure drop between pipe's ends. Such model describes real physical situation and has relevance to some important industrial and engineering applications arising in injection molding, fluid film lubrication, geophysics, etc. Naturally, we cannot hope to obtain the exact solution of the corresponding nonlinear 3D boundary-value problem, so we employ a singular perturbation technique. We introduce the small parameter ε, being the ratio between pipe's thickness and its length, and perform a rigorous asymptotic analysis with respect to ε. Starting from stationary Stokes system and allowing general dependance given by (3), we construct the asymptotic approximation of the governing problem in the form of the explicit formulae and that represents our main contribution. The key idea is to conveniently transform the original problem into the Stokes system with small nonlinear perturbation. Furthermore, we provide a rigorous justification of the formally derived asymptotic approximation by proving the error estimates in the appropriate norm. We believe that the result presented in the present paper provides a good platform for understanding the direct influence of the pressure-dependent viscosity on the pipe flow and, thus, could be instrumental for improving the known engineering practice.

2. Description of the problem

2.1. The pipe's geometry

In order to describe our three-dimensional domain with a small parameter ε appearing explicitly, we first introduce

$$\Omega = \{(x_1, y_2, y_3) \in \mathbf{R}^3 : 0 < x_1 < \ell,\ (y_2, y_3) \in S(x_1)\},$$

where the family of bounded domains $\{S(x_1)\}_{x_1 \in [0,\ell]} \subset \mathbf{R}^2$ is chosen such that Ω is locally Lipschitz. Now we define the thin (or long) pipe with variable cross-section $S(x_1)$ and length $\ell$by

$$\Omega^\varepsilon = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3 : 0 < x_1 < \ell,\ (x_2, x_3) \in \varepsilon\ S(x_1)\}.$$

We denote the ends of the pipe by

$$\Sigma_i^\varepsilon = \varepsilon\ S(i), \quad i = 0, \ell.$$

The lateral boundary of the pipe is given by

$$\Gamma^\varepsilon = \{x = (x_1, x_2, x_3) \in \mathbf{R}^3 : 0 < x_1 < \ell,\ (x_2, x_3) \in \varepsilon\ \partial S(x_1)\}.$$

2.2. The equations and boundary conditions

For an incompressible viscous fluid with a pressure-dependent viscosity, we can assume that the stress tensor is given by

$$\mathbf{T} = -p\mathbf{I} + 2\mu(p)\mathbf{D}(\mathbf{u}),$$

where

$$\mathbf{D}(\mathbf{u}) = \frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^T]$$

is the symmetric part of the velocity gradient. In this case, the stationary Navier–Stokes equations in the absence of external forces become:

$$-\operatorname{div}[2\mu(p)\mathbf{D}(\mathbf{u})] + (\mathbf{u}\nabla)\mathbf{u} + \nabla p = 0,$$
$$\operatorname{div}\mathbf{u} = 0.$$

If the Reynolds number is not too large it is reasonable to neglect the inertial term so we consider the following system:

$$-\operatorname{div}[2\mu(p^\varepsilon)\mathbf{D}(\mathbf{u}^\varepsilon)] + \nabla p^\varepsilon = 0 \text{ in } \Omega^\varepsilon, \tag{4}$$
$$\operatorname{div}\mathbf{u}^\varepsilon = 0 \quad \text{in} \quad \Omega^\varepsilon. \tag{5}$$

The unknown functions are $\mathbf{u}^\varepsilon$ and p^ε representing the velocity and pressure of the fluid. We added the superscript ε in order to stress the dependence of the solution on the small parameter. As emphasized in the Introduction, we suppose that viscosity $\mu(p)$ is a C^1 function such that

$$0 < \mu(\xi) \leqslant C_1\ e^{\alpha\xi}, \quad 0 \leqslant \mu'(\xi) \leqslant C_2\ e^{\alpha\xi},\ \forall\xi$$
$$(C_1, C_2, \alpha = \text{const.} > 0). \tag{6}$$

The flow is assumed to be governed by the prescribed pressure drop. Thus, we impose the following boundary conditions

$$\mathbf{u}^\varepsilon = 0 \quad \text{on} \quad \Gamma^\varepsilon, \tag{7}$$
$$\mathbf{e}_1 \times \mathbf{u}^\varepsilon = 0, \quad p^\varepsilon = p_i \quad \text{on} \quad \Sigma_i^\varepsilon,\ i = 0, \ell, \tag{8}$$

for given pressures p_i, $i = 0$, ℓ ($p_0 > p_\ell$). Here and in the sequel $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ denotes the standard Cartesian basis.

The existence and uniqueness of the solution for such problem was discussed by Marušić-Paloka [18]. The well-posedness is proved under the assumption (6) on the viscosity law and with velocity prescribed on the whole boundary. The only difference here is that we prescribe the pressure drop between pipe's ends in order to treat the situation naturally arising in the applications. Since we neglected the inertial term in the momentum equation, pressure boundary condition does not represent a serious obstacle and, thus, the adaptation of the proof from [18] is straightforward (see also [19]). The goal of the present paper is to find the asymptotic behavior of the flow governed by (4)–(8), as the pipe's thickness $\varepsilon \to 0$.

3. Asymptotic analysis

3.1. Transformation of the problem

The first step is to conveniently transform the governing system. We first explain the procedure assuming that μ satisfies Barus formula

$$\mu(\xi) = \mu_0 e^{\alpha\xi}, \quad \mu_0, \alpha = \text{const.} > 0 \tag{9}$$

and then generalize it for general viscosity-pressure relation obeying (6). Taking into account (9), the momentum equation can be rewritten as

$$0 = -\operatorname{div}[2\mu_0 e^{\alpha p^\varepsilon}\mathbf{D}(\mathbf{u}^\varepsilon)] + \nabla p^\varepsilon$$
$$= -\mu_0 e^{\alpha p^\varepsilon}\Delta\mathbf{u}^\varepsilon - 2\mu_0\alpha\ e^{\alpha p^\varepsilon}\mathbf{D}(\mathbf{u}^\varepsilon)\nabla p^\varepsilon + \nabla p^\varepsilon.$$

Dividing the above equation by $\mu_0 e^{\alpha p^\varepsilon}$ we obtain

$$-\Delta\mathbf{u}^\varepsilon + \frac{1}{\mu_0}\ e^{-\alpha p^\varepsilon}\nabla p^\varepsilon = 2\alpha\ \mathbf{D}(\mathbf{u}^\varepsilon)\nabla p^\varepsilon. \tag{10}$$

That motivates us to introduce a new function $q^\varepsilon = B(p^\varepsilon)$ such that

$$\frac{1}{\mu_0}\ e^{-\alpha p^\varepsilon}\nabla p^\varepsilon = \nabla q^\varepsilon. \tag{11}$$

Obviously, there exists a continuum of such functions given by

$$q^\varepsilon = B(p^\varepsilon) = \frac{1}{\alpha\mu_0}(e^{-\alpha q_0} - e^{-\alpha p^\varepsilon}), \tag{12}$$

with $q_0 \in \mathbf{R}$ being arbitrary.

Remark 1. Assuming that p^ε ranges between p_ℓ and p_0 and that the behavior of the fluid beyond these limits is irrelevant, it follows

$$\frac{1}{\alpha\mu_0}(e^{-\alpha q_0} - e^{-\alpha p_\ell}) \leqslant q^\varepsilon \leqslant \frac{1}{\alpha\mu_0}(e^{-\alpha q_0} - e^{-\alpha p_0}) \tag{13}$$

implying