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# Isogeometric locking-free plate element: A simple first order shear deformation theory for functionally graded plates

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ABSTRACT

An effective, simple, robust and locking-free plate formulation is proposed to analyze the static bending, buckling, and free vibration of homogeneous and functionally graded plates. The simple first-order shear deformation theory (S-FSDT), which was recently presented in Thai and Choi (2013) [11], is naturally free from shear-locking and captures the physics of the shear-deformation effect present in the original FSDT, whilst also being less computationally expensive due to having fewer unknowns. The S-FSDT requires $C^1$-continuity that is simple to satisfy with the inherent high-order continuity of the non-uniform rational B-spline (NURBS) basis functions, which we use in the framework of isogeometric analysis (IGA). Numerical examples are solved and the results are compared with reference solutions to confirm the accuracy of the proposed method. Furthermore, the effects of boundary conditions, gradient index, and geometric shape on the mechanical response of functionally graded plates are investigated.


## 1. Introduction

Functionally graded plates (FG plates) are a special type of composite structures with continuous variation of material properties between the top and bottom surfaces of the plate. Due to the advantageous mechanical behaviors of FG plates they are seeing increased use in a variety of engineering applications [1]. A significant number of studies have been performed to examine the mechanical behavior of FG plates see e.g., [2] for a review. It is widely accepted [2] that plate theories such as the first-order shear deformable theory (FSDT), sometimes also referred to as the Reissner–Mindlin theory, that take into account the shear-deformation effect are necessary to adequately capture the physical behavior of thick plates. Therefore the classical, or Kirchhoff plate theory, which does not capture the effect of shear deformations is not a suitable model for thick FG plates.

Historically the FSDT [3], sometimes also referred to as the Reissner–Mindlin theory, has been popular in computational mechanics for two main reasons: firstly, as mentioned above, it captures the extra physics of shear-deformation not present in the classical theory, and secondly, it relaxes the $C^1$ continuity requirement of the classical theory to $C^0$. This $C^0$ continuity requirement is easier to satisfy using the low-order Lagrangian finite elements that form the basis of most finite element packages. However it is well-known that naïve numerical implementations of the standard FSDT using low-order Lagrangian shape functions typically suffer from shear-locking in the thin-plate or Kirchhoff limit resulting in totally incorrect solutions. Special techniques such as the MITC family of elements [4], assumed strain method [5], field consistent approach [6], smoothed finite element [7] with strain smoothing stabilization technique, are often applied to solve the shear-locking problem, but with additional expense and implementation complexity.

However, with the introduction of numerical methods relying on basis functions with natural $C^1$ continuity such as NURBS in an isogeometric analysis framework (IGA) [8] and meshfree methods [9,10] we believe that the physical accuracy and straightforward numerical implementation are no longer at odds. In this paper we develop a simple, efficient, robust and locking-free numerical method for thin through to thick shear-deformable plates by using a $C^1$ continuity formulation that includes the effects of shear-deformation. We prove its efficacy by studying homogeneous and functionally graded plates.

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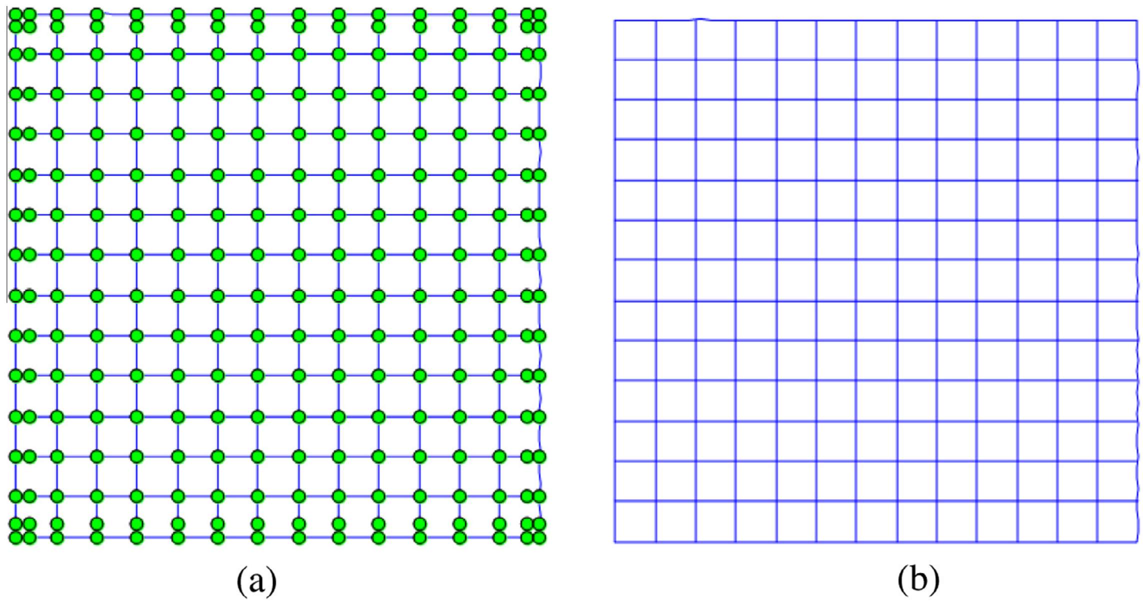


**Fig. 1.** A square plate with 16 × 16 control points and 13 × 13 elements by using cubic NURBS basis function: (a) control mesh and (b) physical mesh.

**Table 1**
Comparisons of the normalized central deflection obtained by the IGA for the S-FSDT, FSDT, and analytical method [35].

| $a/h$ | Method | $\bar{w}$ | Error (%) | $a/h$ | Method | $\bar{w}$ | Error (%) |
|---|---|---|---|---|---|---|---|
| *(a) Fully simply supported* | | | | | | | |
| 20 | FSDT-quadratic | 0.4114 | 0.1461 | 100 | FSDT-quadratic | **0.4048** | 0.3937 |
| | S-FSDT-quadratic | 0.4109 | 0.0243 | | S-FSDT-quadratic | 0.4058 | 0.1476 |
| | FSDT-cubic | 0.4115 | 0.1704 | | FSDT-cubic | 0.4064 | 0 |
| | S-FSDT-cubic | 0.4115 | 0.1704 | | S-FSDT-cubic | 0.4064 | 0 |
| | Exact [35] | 0.4108 | | | Exact [35] | 0.4064 | |
| $10^3$ | FSDT-quadratic | 0.3654 | 10.0443 | $10^4$ | FSDT-quadratic | 0.3552 | 12.5554 |
| | S-FSDT-quadratic | 0.4056 | 0.1477 | | S-FSDT-quadratic | 0.4056 | 0.1477 |
| | FSDT-cubic | **0.4069** | 0.1723 | | FSDT-cubic | 0.3825 | 5.8346 |
| | S-FSDT-cubic | 0.4062 | 0 | | S-FSDT-cubic | 0.4062 | 0 |
| | Exact [35] | 0.4062 | | | Exact [35] | 0.4062 | |
| $10^5$ | FSDT-quadratic | 0.3551 | 12.58 | $10^6$ | FSDT-quadratic | 0.3551 | 12.58 |
| | S-FSDT-quadratic | 0.4056 | 0.1477 | | S-FSDT-quadratic | 0.4056 | 0.1477 |
| | FSDT-cubic | 0.3556 | 12.4569 | | FSDT-cubic | 0.3551 | 12.58 |
| | S-FSDT-cubic | 0.4062 | 0 | | S-FSDT-cubic | 0.4062 | 0 |
| | Exact [35] | 0.4062 | | | Exact [35] | 0.4062 | |
| *(b) Fully clamped* | | | | | | | |
| 100 | FSDT-quadratic | **0.1213** | 4.1107 | $10^3$ | FSDT-quadratic | 0.0252 | 80.0791 |
| | S-FSDT-quadratic | 0.1244 | 1.6601 | | S-FSDT-quadratic | 0.1242 | 1.8182 |
| | FSDT-cubic | 0.1268 | 0.2372 | | FSDT-cubic | **0.1268** | 0.2372 |
| | S-FSDT-cubic | 0.1267 | 0.1581 | | S-FSDT-cubic | 0.1265 | 0 |
| | Exact [35] | 0.1265 | | | Exact [35] | 0.1265 | |
| $10^4$ | FSDT-quadratic | $3.1 \times 10^{-4}$ | 99.75 | $10^5$ | FSDT-quadratic | $3.2 \times 10^{-6}$ | 100 |
| | S-FSDT-quadratic | 0.1242 | 1.8182 | | S-FSDT-quadratic | 0.1242 | 1.8182 |
| | FSDT-cubic | 0.0674 | 46.7194 | | FSDT-cubic | 0.0014 | 98.8933 |
| | S-FSDT-cubic | 0.1265 | 0 | | S-FSDT-cubic | 0.1265 | 0 |
| | Exact [35] | 0.1265 | | | Exact [35] | 0.1265 | |
| $10^6$ | FSDT -quadratic | $3.2 \times 10^{-8}$ | 100 | | | | |
| | S-FSDT-quadratic | 0.1242 | 1.8182 | | | | |
| | FSDT-cubic | $1.4 \times 10^{-5}$ | 99.99 | | | | |
| | S-FSDT-cubic | 0.1265 | 0 | | | | |
| | Exact [35] | 0.1265 | | | | | |

The underlying differential equation in our formulation is based on the simple FSDT (S-FSDT) recently presented in [11,12]. The key idea in the derivation of the S-FSDT is the decomposition of the transverse displacements in the FSDT into bending and shear parts before eliminating the rotation variables using the partial derivatives of the transverse bending displacement only. This weak formulation of the S-FSDT problem requires $C^1$ continuity just like in the classical plate theory but also includes the shear deformable physics of the FSDT. Therefore as well as being viewed as a simple FSDT, this formulation could also be viewed as a classical theory augmented with the shear-deformable physics of the FSDT formulation. Furthermore because the rotation variables of the standard FSDT are eliminated in terms of the bending transverse displacements the resulting weak formulation contains only four variables