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Three-dimensional elastic deformation of functionally graded isotropic plates under point loading

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ABSTRACT

In this paper, three-dimensional elastic deformation of isotropic functionally graded plates subjected to point loading is investigated using a combination of analytical and computational means. The analytical approach is based on the displacement functions method, while numerical modeling, which requires high accuracy in the representation of the point loading, uses Galerkin type finite element method. Three different plate geometries are examined for validation purposes, and the difficulties associated with an optimum choice of the element size are discussed. It is shown that by using a posteriori error estimation based on the equivalent stress measure accurate results can be obtained even in the neighborhood of the point loading.

1. Introduction

The concept of functionally graded material (FGM) is currently actively explored in a variety of engineering and biomedical applications where conventional materials can no longer meet increased expectations in terms of performance and structural integrity. FGM refers to advanced composite materials with gradual compositional variation of the constituents from one surface of the material to the other, which results in a continuous variation of material properties. A comprehensive review of the principal development in the modeling of functionally graded materials and structures covering homogenization of particulate FGMs, heat transfer, statics, dynamics, stability fracture, testing and design is given by Birman and Byrd [3].

The study of a structure's response to point loading is an important step towards investigation of more complex loading problems including indentation and impact, which requires an understanding of both the local and global response of the material and structure. Much research concentrated on the development of analytical/numerical solutions for isotropic and anisotropic functionally graded half-planes and half-spaces with power-law, exponential and linear variation of the elastic constants with respect to depth, see the survey in [23]. By neglecting boundary dimensions and curvatures, these studies have restricted their interest to the near-field behavior in order to provide a valuable insight into the local stress distribution near the surface of FGMs.

Despite the fact that structures such as beams, plates and shells are frequently subjected to concentrated loads under working or experimental conditions, elastic deformation of functionally graded structures under point loading has received considerably less attention in the literature. Whilst a number of plate theories for functionally graded plates have been proposed (see e.g., [3]), numerical examples that accompany them are usually restricted to one-term sinusoidal loading and uniformly distributed loading.

A solution to the problem of a concentrated line force acting in the interior of an infinite plate was developed by Spencer [19]. The plate was assumed to be of arbitrary thickness, isotropic and inhomogeneous, with the elastic moduli being functions, not necessarily continuous, of the through-thickness coordinate. The mechanical properties of the plate are not necessarily symmetric about the mid-surface. The solution, based on the classical solution for a concentrated force in a thin elastic plate, was extended to give exact closed form solutions for the displacement and stress in the thick inhomogeneous plate.

Guo et al. [8] examined the interface crack problem for an infinite plate of finite thickness with functionally graded coating subjected to a concentrated force. An exponential variation of the shear modulus in the coating was assumed.

More recently, Woodward and Kashtalyan [24] investigated the elastic deformation of rectangular sandwich panels with a graded core subjected to various types of localized loads including patch,

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line, and point load. The elastic behavior of functionally graded rectangular plates under patch load was also studied in Woodward and Kashtalyan [25]. Patches of three different sizes were considered: full-size patch (i.e., uniformly distributed loading), large centralized patch, and small centralized patch. Analytical modeling was based on 3-D elasticity solution for stress and displacement fields in functionally graded plates subjected to a one-term sinusoidal loading recently developed by Kashtalyan [13], while finite element modeling was performed in ABAQUS with user implemented graded elements. Agreement between the 3-D elasticity solution and finite element model was excellent. It is worth noting that while increasing the number of terms used in the Fourier representation of patch load type can be seen to give greater accuracy at the plate center, an overshoot in the normal stress is observed in locations of patch load application as the solution tries to capture a step change in load with number of sinusoidal terms. This is known as the Gibbs's phenomenon ([5,6]).

Singh and Shukla [18] performed a nonlinear flexural analysis of simply supported and clamped functionally graded plates under line and point loadings. They employed the Levinson shear deformation theory and multiquadratic radial basis functions methods to study the effect of stiffness gradient and boundary conditions on central deflection and in-plane stresses in the plates with power-law variation of Young's modulus through the thickness of the plate.

Sun and Luo [21,20,22] investigated wave propagation and transient response of functionally graded plates under a point impact loading, while Doddamani et al. [4] studied the behavior of sandwich beams with functionally graded rubber under three point bending by using a combination of experimental and numerical techniques.

In this paper, the three-dimensional elastic deformation of an FGM rectangular plate subjected to point load is investigated by a combination of analytical and computational tools.

2. Analytical modeling

2.1. Problem formulation

Consider a rectangular plate of length a, width b and thickness h. The plate is a three-dimensional continuous body, $\mathcal{B}_0$, which is referred to the material configuration expressed in a Cartesian co-ordinate system (X_1, X_2, X_3), so that $0 \leqslant X_1 \leqslant a, 0 \leqslant X_2 \leqslant b, 0 \leqslant X_3 \leqslant h$, cf., Fig. 1.

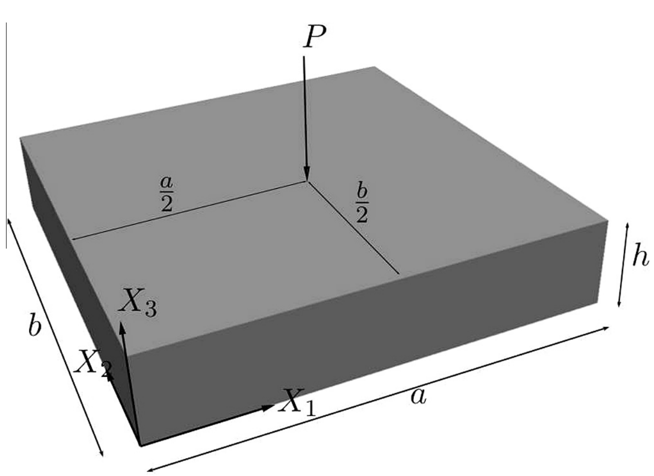

Fig. 1. Geometry and loading of the three-dimensional continuum body $\mathcal{B}_0$.

The material of the FGM plate is assumed to be isotropic inhomogeneous, with an exponential variation of the shear modulus G with the thickness co-ordinate X_3 in the form:

$$G(X_3) = G_1 \exp\left(\gamma\left(\frac{X_3}{h} - 1\right)\right), \qquad \gamma = \ln\left(\frac{G_1}{G_0}\right), \quad \nu = \text{const.}. \tag{1}$$

Here G_0 is the value of the shear modulus at the bottom surface of the plate, $X_3 = 0, G_1$ is the value of the shear modulus at the top surface of the plate, $X_3 = h$, and γ is the inhomogeneity parameter. The Poisson's ratio is assumed to be constant.

If the displacement formulation is used, the three-dimensional displacement field in the plate is governed by the following three equilibrium equations in terms of displacements u_i:

$$\begin{aligned} &G\Delta u_1 + \frac{G}{1-2\nu}\frac{\partial \varepsilon_{kk}}{\partial X_1} + \left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\right)\frac{\mathrm{d}G}{\mathrm{d}X_3} = 0,\\ &G\Delta u_2 + \frac{G}{1-2\nu}\frac{\partial \varepsilon_{kk}}{\partial X_2} + \left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\right)\frac{\mathrm{d}G}{\mathrm{d}X_3} = 0,\\ &G\Delta u_3 + \frac{G}{1-2\nu}\frac{\partial \varepsilon_{kk}}{\partial X_3} + \varepsilon_{kk}\frac{\mathrm{d}}{\mathrm{d}X_3}\left(\frac{2G\nu}{1-2\nu}\right) + 2\frac{\partial u_3}{\partial X_3}\frac{\mathrm{d}G}{\mathrm{d}X_3} = 0, \end{aligned} \tag{2}$$

where and henceforth the summation convention from one to three in repeated indices is applied and the Laplacean operator $\Delta = \frac{\partial^2}{\partial X_i \partial X_i}$ as well as the linearized symmetric strains $\varepsilon_{ij} = \frac{\partial u_{(i}}{\partial X_{j)}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$ are employed in the usual way. Hence ε_{kk} is the volumetric strain or dilatation. The above equations are analogous to the Navier-Lamé equations for homogeneous isotropic materials. In Eq. (2) for the isotropic heterogeneous material or in the Navier-Lamé equations for the isotropic homogeneous case, the same constitutive relation, i.e., Hooke's law defines the Cauchy stress tensor:

$$\sigma_{ij} = \frac{2G\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij}. \tag{3}$$

The plate is subjected to a concentrated (point) force, P, applied at the center of its top surface, $(a/2, b/2, h)$, while the bottom surface remains free, cf., Fig. 1. We will treat the point force as a particular case of distributed transverse loading:

$$Q(X_1, X_2) = P\delta(X_1 - X_1^0)\delta(X_2 - X_2^0), \tag{4}$$

where $\delta(X_1 - X_1^0), \delta(X_2 - X_2^0)$ are Delta-functions such that at $X_1 = X_1^0 = a/2, X_2 = X_2^0 = b/2$ they be of value one and vanish elsewhere. Then the boundary conditions at the top and bottom surfaces of the plate are

$$\begin{aligned} &X_3 = h : \sigma_{33} = Q(X_1, X_2), \sigma_{13} = \sigma_{23} = 0,\\ &X_3 = 0 : \sigma_{33} = \sigma_{13} = \sigma_{23} = 0. \end{aligned} \tag{5}$$

At the edges of the plate, Navier-type boundary conditions are prescribed so that:

$$\begin{aligned} &X_1 = 0, X_1 = a : \sigma_{11} = 0, u_2 = u_3 = 0,\\ &X_2 = 0, X_2 = b : \sigma_{22} = 0, u_1 = u_3 = 0. \end{aligned} \tag{6}$$

These boundary conditions are representative of roller supports and analogous to simply supported edges used in plate theories.

In order to find the analytical solution to Eq. (2) subject to boundary conditions Eqs. (5), (6), we employ Plevako's displacement potential functions $L = \bar{L}(X_i)$ and $N = \bar{N}(X_i)$. The displacement can be represented in terms of the potential functions as:

$$\begin{aligned} &u_1 = -\frac{1}{2G}\left(\nu\Delta - \frac{\partial^2}{\partial X_3^2}\right)\frac{\partial L}{\partial X_1} + \frac{\partial N}{\partial X_2},\\ &u_2 = -\frac{1}{2G}\left(\nu\Delta - \frac{\partial^2}{\partial X_3^2}\right)\frac{\partial L}{\partial X_2} + \frac{\partial N}{\partial X_1}, \end{aligned} \tag{7}$$