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A new Bernoulli–Euler beam model based on a simplified strain gradient elasticity theory and its applications

Xu Liang, Shuling Hu, Shengping Shen *

State Key Laboratory for Strength and Vibration of Mechanical Structures, School of Aerospace, Xi'an Jiaotong University, Xi'an 710049, China

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ABSTRACT

A new Bernoulli–Euler beam model based on a simplified strain gradient elasticity theory is established in the current investigation. The generalized Euler–Lagrange equations and corresponding boundary conditions are naturally derived from the Hamilton's principle. Then axial wave propagation of small scale bars, static bending of cantilever beams, buckling and free vibration of simply supported beams are analytically solved by using the simplified strain gradient beam theory. The influences of the Poisson's effect as well as the weak non-local strain gradient elastic effect are discussed. The Poisson's effect is found to increase with the increase of the beam thickness in the buckling analysis, while the higher-order bending moment induced by stretch strain gradient has an insignificant influence on the critical buckling load in our numerical analysis. However, the effect of the higher-order bending moment is very significant on axial wave propagation and static bending of micro-scale beams. The current work is very helpful in understanding the microstructure-related size dependent phenomenon.

1. Introduction

The lacking of the material length scale parameter in the classical continuum elasticity theory leads to an inaccurate description of the structural behavior at micron and nanometer scale [1]. Higher-order elastic theories such as elasticity theory with couple stresses [2,3], strain gradient elasticity theory [4,5], micropolar elasticity theory [6] and non-local elasticity theory [7] have been established to describe material and structural behaviors with micro-structure. Among these higher-order elasticity theories, the linear strain gradient elasticity theory has been extensively investigated both in plastic [8–11] and elastic domains [12–30]. The linear strain gradient elasticity theory involves additional material length scale parameters besides the classical material constants so that it can be used to determine the size dependent phenomenon at small scale structures.

Mindlin was one of the pioneers who firstly posed the linear elasticity theory with micro-structure [4]. Subsequently Mindlin summarized the linear strain gradient elasticity for isotropic materials, in which 16 additional material length scale parameters are included besides two classical Lamé constants [5]. In this linear strain gradient elasticity theories, the strain energy density depends on both the classical strain and the gradients of strain [5]. On the basis of Minlin's linear strain gradient elasticity theory, a class of phenomenological strain gradient plasticity theory is formulated [11] to account for the strain gradient effects at the micron scale. Lam et al. [14] rewrote the linear strain gradients elasticity theory [4,11] and experimentally determined the material parameters of epoxy cantilever from a bending procedure. A modified couple stress elasticity theory (MCSET) was meanwhile proposed based on the generalized linear strain gradient elasticity theory (SGET) [31]. Park and Gao [32] and Kong et al. [16,21] developed Bernoulli–Euler beam models to incorporate the strain gradients effects at small scale structures. However, the constitutive formulation of the general strain gradient elasticity theory is extremely complex and difficulty in analysis of structural behaviors. The difficulties in solving the boundary value problems in strain gradient elasticity theory, even in the simple models are very complex. A simple form strain gradient elasticity theory is novel and inviting for analyzing structural behavior with consideration of microstructural effect, which is essential because of the widely applications of such structures in micro-systems. A simplified strain gradient elasticity theory (SSGET) was thereby proposed by Altan and Aifantis [33] to formulate a simple linear strain gradient elasticity theory. Recently, Gao and Park [34] proposed the variational formulation of the simplified strain gradient elasticity theory and directly applied the obtained displacement form into the problem of a pressurized thick-walled cylinder and spherical shell [35]. Bending and buckling problems of thin elastic beams with the strain gradients were analyzed by Lazopoulos and Lazopoulos [22] based on the simplified strain gradient elasticity theory with the surface energy. The linear and non-linear plate models based on the simplified strain gradient elasticity theory were also proposed to show

* Corresponding author. Tel./fax: +86 29 82660977.
E-mail address: sshen@mail.xjtu.edu.cn (S. Shen).

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the size dependent phenomena at small scale structures [18,26]. The non-linear micro-beam model based on the simplified strain gradient elasticity theory with surface energy [28] and the non-linear functionally graded micro-beam models [27–30,36] were developed to study the size dependent structural behavior. The general linear strain gradient elasticity theory has been extensively applied to solve the static and dynamic problems of micro-beams [12,13,15,17,18,20,22,23,29,37,38]. However, all the aforementioned studies assumed a uniaxial stress condition and the effect of the Poisson's ratio was neglected. The microstructure-dependent Timoshenko beam model based on the modified couple stress theory proposed by Ma et al. [39] showed that both the effects of couple stress and the Poisson's ratio are very significant for small scale structures. The micro-structure dependent Bernoulli–Euler beam model and Timoshenko beam model for functionally graded beams proposed by Reddy [36,40] also indicate the significance of the geometric non-linearity and Poison's effect. It should be noted that the matrix of the sixth order strain gradient elastic tensor for anisotropic cases is very complex and the matrix representation for all the 3D anisotropic cases was recently discussed by Auffray et al. [41]. For anisotropic materials and the electromechanically coupling issues [42], Hu and Shen [43] developed a theory for nano-dielectrics with the electric field gradient and surface effects, Shen and Hu [44] developed a theory for nano-dielectrics with the flexoelectricity and surface effects. Liang et al. [45] established a simple Bernoulli–Euler dielectric beam model with the strain gradient effect for electromechanical coupling problems. These aforementioned works showed that the effect of the strain gradient elasticity is very significant when the thickness of a beam is very small.

The objection of the present study is to provide a variational formulation of the simplified linear strain gradient beam theory by applying the Euler beam assumption, and the wave propagation, static bending, buckling and free vibration of micro-beams with microstructural effect are analytically solved. The non-linearity, the Poisson's ratio and the strain gradient elastic effects are quantitatively determined.

2. Theory and beam model

2.1. The simplified strain gradient elasticity theory (SSGET)

The strain energy density function for isotropic linear elastic materials in the simplified strain gradient elasticity theory can be expressed as the quadratic of the classical strain and its first gradients [34]

$$u = u(\varepsilon_{ij}, \kappa_{ijk}) = \mu\varepsilon_{ij}\varepsilon_{ij} + \frac{\lambda}{2}\varepsilon_{ii}\varepsilon_{jj} + l^2\left(\mu\kappa_{ijk}\kappa_{ijk} + \frac{\lambda}{2}\kappa_{iik}\kappa_{jjk}\right) \tag{1}$$

where μ and λ are the Lamé constants in the classical elasticity theory. l is a length scale parameter which corresponds to the strain gradient elasticity. ε_{ij} is the classical strain tensor and κ_{ijk} is the third rank strain gradient tensor, which are defined, respectively, as

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \kappa_{ijk} = \varepsilon_{ij,k} = \frac{1}{2}(u_{i,jk} + u_{j,ik}) \tag{2}$$

Thus, one has $\varepsilon_{ij} = \varepsilon_{ji}$, $\kappa_{ijk} = \kappa_{jik}$.

Under the infinitesimal deformation, the constitutive equations for an isotropic linear elastic material can be obtained from the strain energy density as

$$\sigma_{ij} = \frac{\partial u}{\partial \varepsilon_{ij}} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{ll}\delta_{ij}, \quad \tau_{ijk} = \frac{\partial u}{\partial \kappa_{ijk}} = l^2(2\mu\kappa_{ijk} + \lambda\kappa_{llk}\delta_{ij}) \tag{3}$$

where σ_{ij} is the classical Cauchy stress tensor and τ_{ijk} is the higher-order stress (moment stress or double stress) tensor. It is noted that $\sigma_{ij} = \sigma_{ji}$, $\kappa_{ijk} = \kappa_{jik}$.

By means of the Eqs. (1) and (3), the strain energy density can be expressed as

$$u = u(\varepsilon_{ij}, \kappa_{ijk}) = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} + \frac{1}{2}\tau_{ijk}\kappa_{ijk} \tag{4}$$

2.2. Bernoulli–Euler beam model

The simple Bernoulli–Euler beam model which only takes account into the axial deformation and neglects the shear deformation is usually suitable for slender beams. The classical Bernoulli–Euler beam theory assumes that the beam thickness is much less than the radius of curvature induced by external loading and the cross-section of the beam is constant along the length of the beam. The coordinate system is often chosen as x-axis is along the beam length and coherent with the undeformed beam, y-axis points the wide direction and z-axis is along the thickness direction (Fig. 1). In addition, the applied loads and geometry are assumed that the displacement is only functions of x and z coordinates and time t. The component of the displacement along the wide direction is secondary and neglected in the beam theory.

The general expression of the displacement components of Bernoulli–Euler beam can be written as [32,36]

$$u_1 = U_0(x,t) + z\theta(x,t), \quad u_2 = 0, \quad u_3 = W^E(x,t) \tag{5}$$

where u_1, u_2, u_3 are x-, y- and z-components of the displacement vector $\mathbf{u}$, and U_0, W^E are the mid-plane displacement components, $\theta \approx -\frac{\partial W^E}{\partial x}$ is the rotation angle of the cross-section.

The nontrivial strains and strain gradients are obtained from Eqs. (2) and (5) as

$$\varepsilon_{11} = \frac{\partial U_0}{\partial x} - z\frac{\partial^2 W^E}{\partial x^2}$$

$$\kappa_{111} = \frac{\partial^2 U_0}{\partial x^2} - z\frac{\partial^3 W^E}{\partial x^3}, \quad \kappa_{113} = -\frac{\partial^2 W^E}{\partial x^2} \tag{6}$$

The variation of the kinetic energy is

$$\begin{aligned}\delta K &= \int_0^L \int_A \rho\left[\frac{\partial u_1}{\partial t}\frac{\partial \delta u_1}{\partial t} + \frac{\partial u_3}{\partial t}\frac{\partial \delta u_3}{\partial t}\right]\mathrm{d}A\mathrm{d}x \\ &= \int_0^L \left[I_0\left(\frac{\partial U_0}{\partial t}\frac{\partial \delta U_0}{\partial t} + \frac{\partial W^E}{\partial t}\frac{\partial \delta W^E}{\partial t}\right)\right]\mathrm{d}x \\ &\quad - \int_0^L \left[I_1\left(\frac{\partial U_0}{\partial t}\frac{\partial^2 \delta W^E}{\partial x\partial t} + \frac{\partial^2 W^E}{\partial x\partial t}\frac{\partial \delta U_0}{\partial t}\right)\right]\mathrm{d}x \\ &\quad + \int_0^L \left[I_2\frac{\partial^2 W^E}{\partial x\partial t}\frac{\partial^2 \delta W^E}{\partial x\partial t}\right]\mathrm{d}x\end{aligned} \tag{7}$$

where ρ is the mass density and (I_0, I_1, I_2) are the mass inertias

$$(I_0, I_1, I_2) = \int_A \rho(1, z, z^2)\mathrm{d}A \tag{8}$$

The virtual work done by the external forces are

$$\delta W = \int_0^L \left(f\delta U_0 + q\delta W^E + N^E\frac{\partial W^E}{\partial x}\frac{\partial \delta W^E}{\partial x}\right)\mathrm{d}x \tag{9}$$

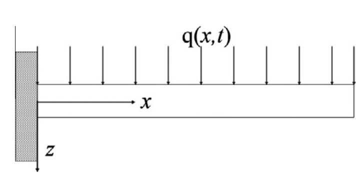

Fig. 1. Cantilever beam configuration.