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Exact analytic solutions of the lubrication equations for squeeze-flow of a biviscous fluid between two parallel disks

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ABSTRACT

Based on lubrication approximation, the squeezing flow of biviscous fluids between two parallel disks with partial slip boundary condition was investigated. In addition to the solution of the kinematics in the bi-viscosity region leading to a cubic equation of the yield surface, the full explicit expressions of radial pressure gradient, pressure and squeeze force are given in the exact relationships. According to three dimensionless numbers, different behaviors are covered including Bingham fluid as a limiting case of bi-viscosity model. Besides, a critical force separating the Newtonian and biviscous regions of the flow is provided. However, for a flow of a Bingham fluid without wall slip, the expression of the applied force may be expressed or not according to the yield stress. This depends on the ratio value of the characteristic time of the fluid to the time scale of observation if it is very lower or higher than unity.

1. Introduction

Since the study of the squeezing flow of fluid between parallel disks has been of great interest due to its wide application in engineering practice, several authors are concerned in its theoretical development. For example, Stefan [1] (quoted in Ref. [2]) can be considered as the first to establish an exact solution for the squeezing flow of viscous Newtonian fluid using the "lubrication approximation". Scott [3] (quoted in Ref. [2]) extended such development to power-law fluids using a no-slip boundary condition between the wall and the fluid. As for Laun et al. [4], they studied the partial slip squeeze flow of Newtonian and power-law fluids using the lubrication approach. It is worth mentioning that the slip condition had already been considered in the classical solution of Stefan. The no-slip squeeze flow of yield stress fluids was originally examined by Scott [3] and Peek [5] (quoted in Ref. [6]) with different rigid cores in the flow region. Later, Covey and Stanmore [7] gave simplified solutions for Bingham and Herschel–Bulkley fluids using lubrication theory. Besides, Meeten [8] carried out both theoretical and experimental investigations of the squeeze flow of Herschel–Bulkley fluid, but he did not account for the slip at the walls. Taking into consideration the partial slip at the plate-sample interface, Sherwood and Durban [9,10] studied the squeeze flow of Bingham and Herschel–Bulkley fluids using asymptotic expansions. With regard to Lawrence and Corfield [11] and Adams et al. [12], they proposed analytical expressions using a slip law which in turn had an "interfacial" yield stress, whose fluids are admitted to behave like a solid in a unidirectional flow. Lipscomb and Denn [13] pointed out that, generally, complex flows cannot admit unyielded zones. According to them, there is a difficulty pertaining to the existence of yield surfaces in the interior of the fluid which must be solid and cannot move along the radial direction, simply because this is kinematically impossible. Other researchers [13–15] advocated that the Bingham model can be viewed as the limiting case of a biviscous fluid and therefore the squeeze flow paradox counteracts. They analyzed the problem using lubrication approximation with no-slip boundary conditions at the walls. Recently, Yang and Zhu [16] have theoretically analyzed the squeeze flow of the Bingham material described by the bi-viscosity model in the small gap between parallel disks with the Navier slip condition. They approximated the pressure gradient by a linear function and obtained an approximate expression for the squeeze force.

The present paper undertakes the theoretical study of squeeze flow between two parallel disks of a Bingham fluid, described by the bi-viscosity model with a partial slip at the wall. The exact solution of squeeze force is obtained by means of lubrication theory. Actually, to the best of author's knowledge, there is no exact solution for this problem available in the open literature. Then, the effects of fluid properties, disks velocity, gap width and slip coefficient upon pressure, pressure gradient and squeeze force are presented according to the three dimensionless variables (viscosity ratio, slip coefficient and yield number).

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The paper is organized as follows. First, the mathematical models are presented in Section 2. Afterwards, the analytical solutions are detailed in Section 3. Next, the results are discussed in Section 4. Finally, concluding remarks are given in Section 5.

2. Mathematic models

The flow system shown in Fig. 1 consists of the squeeze flow of a fluid within a narrow gap between two parallel disks. The plates of radius R_a are separated by $2h$ and translated towards each other with a relative velocity $2\dot{h}$. This squeezing speed is obtained by the squeeze force F applied to the disks. The governing differential equations are written in cylindrical polar coordinate system(r, θ, z). For symmetry reasons, the dependence on coordinate θ terms can be omitted. Besides, for a small aspect ratio $h/R_a \ll 1$, the pressure may be considered as constant in the z-direction and the lubrication theory can be used to balance pressure and shear forces.

A quasi-steady state is assumed and the gravitational and inertial effects are neglected. Hence the dynamic equation describing the motion may be written as follows:

$$\frac{\partial \tau_{rz}}{\partial z} = \frac{dp}{dr} \tag{1}$$

where τ_{rz} is the component of deviatoric stress tensor and p is the pressure.

The integration of Eq. (1) with respect to z, using $\tau_{rz} = 0$ at $z = 0$, gives

$$\tau_{rz} = \frac{dp}{dr} z \tag{2}$$

The continuity may be expressed by

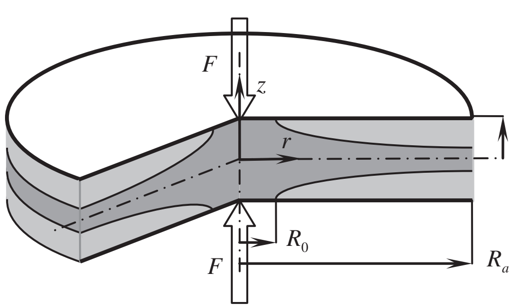

Fig. 1. Schematic diagram of the squeeze flow between two parallel disks (dark region is the core of the fluid with high viscosity).

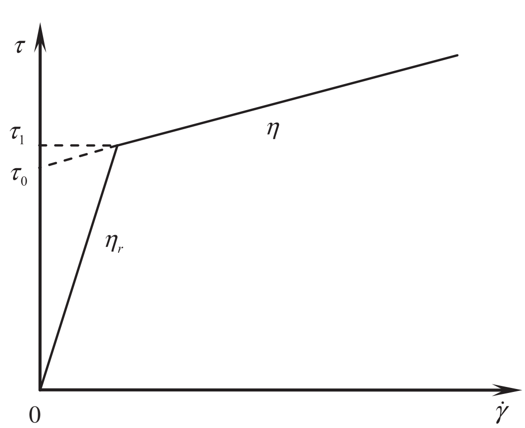

Fig. 2. Model for the biviscous fluid.

$$-\frac{r\dot{h}}{2} = \int_0^h u.dz \tag{3}$$

where $u = u(r,z)$ is the fluid radial velocity.

The constitutive equations of the fluid are those of the bi-viscosity model (Fig. 2) that can be described by

$$\tau_{rz} = \eta_r \frac{\partial u}{\partial z}, \quad |\tau_{rz}| \leqslant \tau_1 \tag{4a}$$

$$\tau_{rz} = sign\left(\frac{\partial u}{\partial z}\right)\tau_0 + \eta \frac{\partial u}{\partial z}, \quad |\tau_{rz}| > \tau_1 \tag{4b}$$

where η_r is the viscosity of the unyielded fluid, η is the viscosity of the yielded fluid, $\tau_0 = \tau_1(1 - \gamma)$ is the yield stress of Bingham fluid, τ_1 is the yield stress of biviscous fluid and $\gamma = \frac{\eta}{\eta_r}$ is the viscosity ratio.

The Bingham fluid can be obtained as a limiting case of bi-viscosity model when $\gamma \to 0$. Some authors [15,17] consider that Newtonian fluid is recovered when $\gamma = 1$ (i.e. $\tau_0 = 0$). where, a priori, $\tau_1 > 0$ can take any value. This proposal does not sufficient, because Newtonian fluid can be recovered when $\gamma \approx 1$ or $\gamma \to 1$ (i.e. $\frac{\tau_0}{\tau_1} \ll 1$).

In the case of the analysis of the fluid squeezing flow with slip at the walls, the slip condition is given by:

$$u = -\beta \tau_{rz}(z = h) = -\beta \frac{dp}{dr} h, \tag{5}$$

where β is the slip coefficient.

While the situation in which $\beta = 0$ corresponds to no slip, full lubrication is described in the limit $\beta \to \infty$. It is worth noting that the slip coefficient should be quite small to be able to consider the lubrication approach.

The flow field is divided into a Newtonian region with high viscosity and a bi-viscosity region with yielded/unyielded fluids, intersecting the disks at $r = R_0$ (Fig. 1).

In the first stage of this development, the dimensional expressions of radial velocity and continuity equation for both regions will be given. In the second stage, the dimensionless expressions of the pressure gradient, pressure and squeeze force will be developed according to three dimensionless parameters which are the viscosity ratio, slip coefficient and yield number.

3. Analytical solutions

A Newtonian region of $r \leqslant R_0$ and a bi-viscosity region of $r > R_0$ are separately considered with different solutions.

3.1. Dimensional development

3.1.1. Kinematics in Newtonian region

In the Newtonian region, the fluid can be considered as a Newtonian fluid with high viscosity, whose constitutive relation is described by Eq. (4a). The combination of Eqs. (1) and (4a) gives

$$\eta_r \frac{\partial^2 u}{\partial z^2} = \frac{dp}{dr} \tag{6}$$

Two integrations with respect to z, using the fact that $\frac{\partial u}{\partial z} = 0$ at $z = 0$, Eq. (5) at $z = h$ and (4a), give the following expression:

$$u = \frac{1}{\eta_r} \frac{dp}{dr} \left[\frac{z^2 - h^2}{2} - \beta \eta_r h \right] \tag{7}$$

Substituting Eq. (7) into Eq. (3) leads to the expression of the pressure gradient

$$\frac{dp}{dr} = \frac{3}{2} \frac{\eta_r \dot{h} r}{h^2 (3\beta\eta_r + h)} \tag{8}$$

The full expressions of the radial velocity and shear stress can be obtained by the substitution of Eq. (8) into Eqs. (7) and (2), respectively