Composite Structures 106 (2013) 33–46

Contents lists available at SciVerse ScienceDirect

Composite Structures

journal homepage: www.elsevier.com/locate/compstruct

A new approach to three-dimensional exact solutions for functionally graded piezoelectric laminated plates

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ARTICLE INFO

Article history:
Available online 6 June 2013

Keywords:
Functionally graded piezoelectric plate
Electroelasticity
3D exact solutions
Sampling surfaces method

ABSTRACT

A paper presents the sampling surfaces (SaS) method and its implementation for the three-dimensional (3D) exact analysis of functionally graded (FG) piezoelectric laminated plates. According to this method, we introduce inside the nth layer I_n not equally spaced SaS parallel to the middle surface of the plate and choose displacement vectors and electric potentials of these surfaces as basic plate variables. Such choice of unknowns with the consequent use of Lagrange polynomials of degree $I_n - 1$ in the thickness direction for each layer leads to a very compact form of governing equations of the FG piezoelectric plate formulation. This fact gives an opportunity to derive the 3D exact solutions of electroelasticity for thick and thin FG piezoelectric laminated plates with a specified accuracy utilizing a sufficient number of SaS, which are located at interfaces and Chebyshev polynomial nodes.

1. Introduction

In recent years, a considerable work has been carried out on the three-dimensional (3D) exact analysis of piezoelectric laminated plates. In the literature, there are at least four approaches to 3D exact solutions of electroelasticity for piezoelectric laminated plates, namely, the Pagano approach, the state space approach, the asymptotic approach and the sampling surfaces (SaS) approach. The first approach [1,2] was applied to piezoelectric plates by Ray et al. [3], Heyliger [4,5], Heyliger and Brooks [6]. The 3D exact analysis of piezoelectric orthotropic and anisotropic plates based on the state space approach was carried out in contributions [7–11]. The asymptotic approach was utilized for derivation of 3D exact solutions for piezoelectric plates [12–15]. The SaS approach was recently implemented for the 3D exact analysis of piezoelectric laminated orthotropic and anisotropic plates [16].

Nowadays, the functionally graded (FG) piezoelectric materials are widely used in mechanical engineering due to their advantages compared to traditional piezoelectric laminated materials. However, the study of FG piezoelectric structures is not a simple task [17] because the material properties depend on the thickness coordinate and some specific assumptions concerning their variations in the thickness direction are required [18,19]. In practice, this implies that we deal here with a system of differential equations with variable coefficients. Therefore, the first two approaches, i.e., the Pagano approach and the state space approach cannot be applied directly to 3D exact solutions for FG piezoelectric plates without using above specific assumptions [20]. On the contrary, the asymptotic approach [21] and the SaS approach can be applied directly to 3D solutions for FG piezoelectric plates because governing differential equations are obtained through definite integration in the thickness direction.

The present paper is intended to show that the SaS method can be also applied efficiently to 3D exact solutions of electroelasticity for FG piezoelectric laminated plates. According to this method, we choose inside the nth layer I_n not equally spaced SaS $\Omega^{(n)1}, \Omega^{(n)2}, \ldots, \Omega^{(n)I_n}$ parallel to the middle surface of the plate and introduce the displacement vectors $\mathbf{u}^{(n)1}, \mathbf{u}^{(n)2}, \ldots, \mathbf{u}^{(n)I_n}$ and the electric potentials $\varphi^{(n)1}, \varphi^{(n)2}, \ldots, \varphi^{(n)I_n}$ of these surfaces as basic plate variables, where $I_n \geqslant 3$. Such choice of unknowns in conjunction with the use of Lagrange polynomials of degree $I_n - 1$ in the thickness direction permits the presentation of governing equations of the proposed FG plate formulation in a very compact form. Note that the SaS method has been already applied to the 3D analysis of elastic and piezoelectric laminated plates and shells [16,22–25].

It should be mentioned that the developed approach with equally spaced SaS [22] does not work properly with Lagrange polynomials of high degree because the Runge's phenomenon can occur, which yields the wild oscillation at the edges of the interval when the user deals with any specific functions. If the number of equally spaced nodes is increased then the oscillations become even larger. Fortunately, the use of Chebyshev polynomial nodes [26] inside each layer can help to improve significantly the behavior of Lagrange polynomials of high degree for which the error will go to zero as $I_n \to \infty$.

An idea of using the SaS can be traced back to [27,28] in which three, four and five equally spaced SaS are employed. These

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http://dx.doi.org/10.1016/j.compstruct.2013.05.037

contributions describe the SaS concept applied to the approximate solution of 3D plate/shell problems. For further information the reader refers to fundamental works [29,30] where the Legendre polynomials in the thickness direction are utilized. However, the use of Legendre polynomials cannot provide a *uniform convergence* of computational procedures to be developed. On the contrary, the SaS method leads to a uniform convergence, as shown in Section 5, that in turn gives an opportunity to derive the 3D exact solutions for FG piezoelectric laminated plates with a prescribed accuracy employing the sufficient number of SaS.

The authors restrict themselves to finding *five right digits* in all examples presented. The better accuracy is possible of course but requires more SaS inside the plate body to be taken.

2. Description of electric field

Consider a FG piezoelectric laminated plate of the thickness h. Let the middle surface Ω be described by Cartesian coordinates x_1 and x_2. The coordinate x_3 is oriented in the thickness direction. The transverse coordinates of SaS inside the nth layer are defined as

$$x_3^{(n)1} = x_3^{[n-1]}, \quad x_3^{(n)I_n} = x_3^{[n]},$$
$$x_3^{(n)m_n} = \frac{1}{2}\left(x_3^{[n-1]} + x_3^{[n]}\right) - \frac{1}{2}h_n \cos\left(\pi\frac{2m_n-3}{2(I_n-2)}\right), \tag{1}$$

where $x_3^{[n-1]}$ and $x_3^{[n]}$ are the transverse coordinates of layer interfaces $\Omega^{[n-1]}$ and $\Omega^{[n]}$ (Fig. 1); $h_n = x_3^{[n]} - x_3^{[n-1]}$ is the thickness of the nth layer; the index n identifies the belonging of any quantity to the nth layer and runs from 1 to N, where N is the number of layers; the index m_n identifies the belonging of any quantity to inner SaS of the nth layer and runs from 2 to $I_n - 1$, whereas the indices i_n, j_n, k_n to be introduced later for describing all SaS of the nth layer run from 1 to I_n.

Remark 1. It is worth noting that transverse coordinates of inner SaS (1) coincide with coordinates of Chebyshev polynomial nodes [26]. This fact has a great meaning for a convergence of the SaS method [23–25].

The relation between the electric field vector and the electric potential φ is given by

$$E_i = -\varphi_{,i}. \tag{2}$$

Here, and in the following developments, indices i, j, k, ℓ range from 1 to 3, whereas Greek indices α, β range from 1 to 2.

The electric field vector at SaS of the nth layer is written as

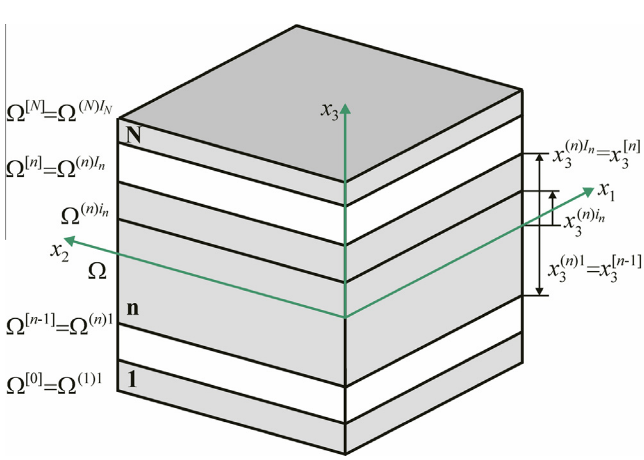

Fig. 1. Geometry of the laminated plate.

$$E_\alpha^{(n)i_n} = E_\alpha\left(x_3^{(n)i_n}\right) = -\varphi_{,\alpha}^{(n)i_n}, \tag{3}$$

$$E_3^{(n)i_n} = E_3\left(x_3^{(n)i_n}\right) = -\psi^{(n)i_n}, \tag{4}$$

where $\varphi^{(n)i_n}(x_1, x_2)$ are the electric potentials of SaS of the nth layer; $\psi^{(n)i_n}(x_1, x_2)$ are the values of the derivative of the electric potential with respect to thickness coordinate at SaS, that is,

$$\varphi^{(n)i_n} = \varphi\left(x_3^{(n)i_n}\right), \quad \psi^{(n)i_n} = \varphi_{,3}(x_3^{(n)i_n}). \tag{5}$$

Next, we assume that the electric potential and the electric field vector are distributed through the thickness of the nth layer as follows:

$$\varphi^{(n)} = \sum_{i_n} L^{(n)i_n}\varphi^{(n)i_n}, \quad x_3^{[n-1]} \leqslant x_3 \leqslant x_3^{[n]}, \tag{6}$$

$$E_i^{(n)} = \sum_{i_n} L^{(n)i_n}E_i^{(n)i_n}, \quad x_3^{[n-1]} \leqslant x_3 \leqslant x_3^{[n]}, \tag{7}$$

where $L^{(n)i_n}(x_3)$ are the Lagrange polynomials of degree $I_n - 1$ expressed as

$$L^{(n)i_n} = \prod_{j_n \neq i_n} \frac{x_3 - x_3^{(n)j_n}}{x_3^{(n)i_n} - x_3^{(n)j_n}}. \tag{8}$$

The use of (5) and (6) leads to a simple formula

$$\psi^{(n)i_n} = \sum_{j_n} M^{(n)j_n}(x_3^{(n)i_n})\varphi^{(n)j_n}, \tag{9}$$

where $M^{(n)j_n} = L_{,3}^{(n)j_n}$ are the derivatives of Lagrange polynomials, which are calculated at SaS of the nth layer as

$$M^{(n)j_n}\left(x_3^{(n)i_n}\right) = \frac{1}{x_3^{(n)j_n} - x_3^{(n)i_n}} \prod_{k_n \neq i_n, j_n} \frac{x_3^{(n)i_n} - x_3^{(n)k_n}}{x_3^{(n)j_n} - x_3^{(n)k_n}} \quad \text{for } j_n \neq i_n,$$
$$M^{(n)i_n}\left(x_3^{(n)i_n}\right) = -\sum_{j_n \neq i_n} M^{(n)j_n}\left(x_3^{(n)i_n}\right). \tag{10}$$

This implies that the key functions $\psi^{(n)i_n}$ of the electric field formulation are represented as a *linear combination* of electric potentials of SaS of the nth layer $\varphi^{(n)j_n}$.

3. Kinematic description of FG laminated plate

The strain tensor is given by

$$2\varepsilon_{ij} = u_{i,j} + u_{j,i}, \tag{11}$$

where u_i are the displacements of the plate. In particular, the strain components at SaS are

$$2\varepsilon_{\alpha\beta}^{(n)i_n} = 2\varepsilon_{\alpha\beta}\left(x_3^{(n)i_n}\right) = u_{\alpha,\beta}^{(n)i_n} + u_{\beta,\alpha}^{(n)i_n},$$
$$2\varepsilon_{\alpha3}^{(n)i_n} = 2\varepsilon_{\alpha3}\left(x_3^{(n)i_n}\right) = \beta_\alpha^{(n)i_n} + u_{3,\alpha}^{(n)i_n}, \tag{12}$$
$$\varepsilon_{33}^{(n)i_n} = \varepsilon_{33}\left(x_3^{(n)i_n}\right) = \beta_3^{(n)i_n},$$

where $u_i^{(n)i_n}(x_1, x_2)$ are the displacements of SaS of the nth layer; $\beta_i^{(n)i_n}(x_1, x_2)$ are the values of derivatives of displacements with respect to coordinate x_3 at SaS, that is,

$$u_i^{(n)i_n} = u_i\left(x_3^{(n)i_n}\right), \quad \beta_i^{(n)i_n} = u_{i,3}\left(x_3^{(n)i_n}\right). \tag{13}$$

The following step consists in a choice of consistent approximation of displacements and strains through the thickness of the nth layer. It is apparent that displacement and strain distributions should be chosen similar to electric field distributions (6) and (7):

$$u_i^{(n)} = \sum_{i_n} L^{(n)i_n}u_i^{(n)i_n}, \quad x_3^{[n-1]} \leqslant x_3 \leqslant x_3^{[n]}, \tag{14}$$