Composite Structures 106 (2013) 74–84

Contents lists available at SciVerse ScienceDirect

Composite Structures

journal homepage: www.elsevier.com/locate/compstruct

# Postbuckling analysis of variable angle tow plates using differential quadrature method

Gangadharan Raju, Zhangming Wu, Paul M. Weaver *

*Advanced Composite Centre for Innovation and Science, Department of Aerospace Engineering, University of Bristol, Queens Building, Bristol BS8 1TR, UK*

ARTICLE INFO

*Article history:*
Available online 23 May 2013

*Keywords:*
Postbuckling
Variable angle tow composites
Airy's stress function
Differential quadrature method

ABSTRACT

Variable angle tow (VAT) placement has been shown to improve the buckling resistance of composite plates under compression loading. The problem of tailoring the in-plane tow path of VAT composite plates for enhanced postbuckling resistance is studied in this work. A pair of coupled geometrically nonlinear governing differential equations in terms of stress function and transverse displacement, based on classical laminated plate theory, is derived for postbuckling analysis of VAT plates. The differential quadrature method (DQM) is applied to solve the differential equations and the resulting nonlinear algebraic equations solved using a Newton–Raphson algorithm. The DQM was applied to study the postbuckling problem of VAT composite plates subjected to axial compression under different plate boundary conditions. The numerical results of DQM are compared with detailed finite element analysis and the relative accuracy and efficiency of the proposed DQM approach is studied.



## 1. Introduction

The buckling performance of variable angle tow composites has been studied extensively and has been shown to exhibit potentially superior structural performance over conventional straight fibre composites [1–3]. The tow steering concept is now considered for improving the postbuckling strength and stiffness of VAT plates. The VAT concept for the design of composite structures in the postbuckling range provides increased tailorability options to reduce weight and improve structural performance. Very few works have been reported on the postbuckling analysis of VAT plates. Rahman et al. [4] studied the postbuckling response of VAT plates using a perturbation approach coupled with finite element modelling. The perturbation approach was used to generate a reduced-order model for computation of postbuckling coefficients to predict the postbuckling stiffness of VAT plates. Biggers et al. [5] used finite elements to study the postbuckling response of piecewise uniform tailored composite plates and demonstrated an improvement in postbuckling stiffness of these plates over uniform composite plates. Lopes et al. [6] studied the buckling and postbuckling failure response characteristics of variable stiffness composites. They used finite element analysis to model to study the failure of VAT plates which requires significant computational effort. In parallel work, we [7,8] proposed a mixed variational approach using stress function and transverse displacement to solve the postbuckling problem of VAT plates. We applied Rayleigh–Ritz method to the variational form and then studied the postbuckling response of square VAT plates under different inplane boundary conditions. Alhajahmad et al. [9] studied the nonlinear pressure pillowing problem using the Rayleigh–Ritz method and showed an improvement in load carrying capacity using VAT panels compared to straight fibre designs. Weaver et al. [10] employed an embroidery machine for manufacture of tow steered plates and showed designs of VAT plates with nonlinear angle distributions. Their experimental and finite element results demonstrated similar buckling performance to a quasi-isotropic composite, but improved postbuckling response. New methods which are fast, accurate and computationally less expensive are required for nonlinear analysis of VAT plates. In this work, a numerical methodology based on the differential quadrature method (DQM) is developed for postbuckling analysis of VAT panels.

The nonlinear behaviour of a plate undergoing large deformation is described using the von Karman strain–displacement relations which are then applied and used with equilibria and constitutive relations to derive the governing partial differential equations. In the literature, various numerical methods like finite element (FE) method, finite strip method, boundary element method and finite difference method have been applied to solve the postbuckling problem of composite plates [11–14]. Further, numerous semi-analytical approaches based on Rayleigh–Ritz, Galerkin and perturbation methods have been proposed to solve the postbuckling problem of composite plates [15–18]. As an alternative to these methods, the DQ approach which is a

* Corresponding author. Tel.: +44 (0) 1173315318.
*E-mail address:* paul.weaver@bristol.ac.uk (P.M. Weaver).

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http://dx.doi.org/10.1016/j.compstruct.2013.05.010

comparatively new method is being investigated for performing nonlinear structural analysis of VAT composite plates. The DQM was previously applied by the authors for prebuckling and buckling analysis of VAT plates [19] and found the approach to be accurate, robust and computationally efficient. Bert et al. [20] studied the nonlinear bending of orthotropic plates using DQM and showed that the method computes solutions of reasonable accuracy with relatively little computational effort. However, the results for simply supported plates were not accurate and this problem was attributed to the numerical procedure used for enforcement of the boundary conditions on the plate edges. To overcome this problem, Chen et al. [21] presented a new DQM approach for applying the boundary conditions and applied special matrix products to formulate nonlinear differential equations into simplified matrix form. Later, Chen successfully applied the new DQ approach to solve the geometrically nonlinear bending problem of isotropic and orthotropic rectangular plates. Li et al. [22] further extended DQM to study the nonlinear bending of orthotropic plates by including the effect of transverse shear on bending deformation. Taheri et al. [23] applied DQM to perform postbuckling analysis of straight fibre composites and used an arclength method to solve the nonlinear algebraic equations. In these works, the nonlinear field equations were written in terms of displacements $(u, v, w)$ and DQM was then applied to solve them. Liew et al. [24] modelled the postbuckling behaviour of functionally graded material plates using the stress function approach and proposed a combined Galerkin-differential quadrature iteration algorithm for solution of the nonlinear field equations. To the authors' knowledge, no other works have been reported in the literature of using DQM alone to solve the postbuckling problem modelled using a stress function approach. In this work, this problem is addressed, i.e. DQM is used to solve the postbuckling problem of straight fibre/VAT rectangular composite plates modelled using Airy's stress function and transverse displacement under axial compression. The advantage of this approach is the use of Airy's stress function to perform the postbuckling analysis of anisotropic composite plates considerably reduces the problem size and when coupled with DQM requires less computational effort than finite element method. The generality of the formulation helps in efficient modelling of pure stress and mixed in-plane boundary conditions applied to the composite plate and also include the effect of flexural-twist coupling coefficients in the postbuckling response of composite plates. The postbuckling performance of VAT panels with linearly varying fibre orientations, in the planform, under different plate boundary conditions is shown using DQM and the results compared with the FE method. The stability and robustness of DQM in computing the postbuckling performance of VAT panels is also analysed.

The remainder of this work is organized as follows. In Section 2, the field equations for postbuckling analysis of VAT composite plates are presented. In Section 3, the numerical aspects of DQM such as grid point distributions, weighting matrices and approaches to apply boundary conditions are discussed. In Section 4, several numerical examples of VAT panels are presented to demonstrate the accuracy of the method and close with a few concluding remarks in Section 5.

## 2. Postbuckling analysis

Variable angle tow placement allows the fibre to be steered along the plane of the plate resulting in stiffness properties varying with inplane coordinates $x$–$y$ of the plate. In the case of symmetric VAT panels, stiffness matrices $A$,$D$ are a function of $x$–$y$ coordinates and the constitutive equation in partially inverse form is given by,

$$\begin{Bmatrix} \epsilon^0 \\ M \end{Bmatrix} = \begin{bmatrix} A^*(x,y) & 0 \\ 0 & D(x,y) \end{bmatrix} \begin{Bmatrix} \overline{N} \\ \kappa \end{Bmatrix} \tag{1}$$

where $\epsilon^0$, $\kappa$ are the midplane strain and curvature, $\overline{N}, M$ are the stress and moment resultants, and $A^* = A^{-1}$ is the compliance matrix. The nonlinear midplane strains $\epsilon^0$ and curvatures $\kappa$ are defined as

$$\begin{aligned} &\epsilon_x^0 = u_{,x} + \frac{1}{2}w_{,x}^2 + w_{,x}w_{0,x}, \quad \epsilon_y^0 = v_{,y} + \frac{1}{2}w_{,y}^2 + w_{,y}w_{0,y}, \\ &\epsilon_{xy}^0 = u_{,y} + v_{,x} + w_{,x}w_{,y} + w_{,x}w_{0,y} + w_{,y}w_{0,x}, \\ &\kappa_x = -w_{,xx}, \quad \kappa_y = -w_{,yy}, \quad \kappa_{xy} = -2w_{,xy} \end{aligned} \tag{2}$$

where $u$, $v$, $w$ are the displacements and $w_0$ is the initial imperfection function [25]. A stress function $\Omega$ is introduced such that the stress resultants are defined by,

$$\overline{N}_x = \Omega_{,yy}, \quad \overline{N}_y = \Omega_{,xx}, \quad \overline{N}_{xy} = -\Omega_{,xy} \tag{3}$$

The compatibility condition in terms of mid-plane strains in a plane stress condition is given by

$$\begin{aligned} \epsilon_{x,yy}^0 + \epsilon_{y,xx}^0 - \epsilon_{xy,xy}^0 = w_{,xy}^2 - w_{,xx}w_{,yy} + 2w_{,xy}w_{0,xy} - w_{,xx}w_{0,yy} \\ - w_{,yy}w_{0,xx} \end{aligned} \tag{4}$$

After substitution of Eqs. (1)–(3) into Eq. (4), the final form is given by

$$\begin{aligned} &A_{11}^*\Omega_{,yyyy} - 2A_{16}^*\Omega_{,xyyy} + (2A_{12}^* + A_{66}^*)\Omega_{,xxyy} - 2A_{26}^*\Omega_{,xxxy} + A_{22}^*\Omega_{,xxxx} \\ &\quad + \left(2A_{11,y}^* - A_{16,x}^*\right)\Omega_{,yyy} + \left(2A_{12,x}^* - 3A_{16,y}^* + A_{66,x}^*\right)\Omega_{,xyy} \\ &\quad + \left(2A_{12,y}^* - 3A_{26,x}^* + A_{66,y}^*\right)\Omega_{,xxy} + \left(2A_{22,x}^* - A_{26,y}^*\right)\Omega_{,xxx} \\ &\quad + \left(A_{11,yy}^* + A_{12,xx}^* - A_{16,xy}^*\right)\Omega_{,yy} + \left(-A_{26,xx}^* - A_{16,yy}^* + A_{66,xy}^*\right)\Omega_{,xy} \\ &\quad + \left(A_{12,yy}^* + A_{22,xx}^* - A_{26,xy}^*\right)\Omega_{,xx} = w_{,xy}^2 - w_{,xx}w_{,yy} + 2w_{,xy}w_{0,xy} \\ &\quad - w_{,xx}w_{0,yy} - w_{,yy}w_{0,xx} \end{aligned} \tag{5}$$

where the terms $A_{ik}^* = A_{ik}^*(x,y);\ i,k = [1,2,6]$ are a function of the $x$–$y$ coordinates and the Eq. (5) represents a fourth order elliptic partial differential equation in terms of stress function with variable coefficients. The differential equation of transverse motion that governs the postbuckling analysis of symmetrical VAT plate is given by,

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + \overline{N}_x\frac{\partial^2 w}{\partial x^2} + 2\overline{N}_{xy}\frac{\partial^2 w}{\partial x \partial y} + \overline{N}_y\frac{\partial^2 w}{\partial y^2} + q = 0 \tag{6}$$

where $M_x$, $M_y$, $M_{xy}$ are the moment distributions and $q$ is the load applied in $z$ direction. Eqs. (1)–(3) are then substituted in Eq. (6) and the resulting differential equation is given by

$$\begin{aligned} &D_{11}w_{,xxxx} + 4D_{16}(x,y)w_{,xxxy} + 2(D_{12} + 2D_{66})w_{,xxyy} + 4D_{26}w_{,yyyx} \\ &\quad + D_{22}w_{,yyyy} + 2(D_{11,x} + D_{16,y})w_{,xxx} + (6D_{16,x} + 2D_{12,y} + 4D_{66,y})w_{,xxy} \\ &\quad + (2D_{12,x} + 4D_{66,x} + 6D_{26,y})w_{,xyy} + 2(D_{26,x} + D_{22,y})w_{,yyy} \\ &\quad + (D_{11,xx} + 2D_{16,xy} + D_{12,yy})w_{,xx} + (2D_{16,xx} + 4D_{66,xy} + 2D_{26,yy})w_{,xy} \\ &\quad + (D_{12,xx} + 2D_{26,xy} + D_{22,yy})w_{,yy} - \Omega_{yy}(w_{,xx} + w_{0,xx}) \\ &\quad + 2\Omega_{xy}(w_{,xy} + w_{0,xy}) - \Omega_{xx}(w_{,yy} + w_{0,yy}) + q = 0 \end{aligned} \tag{7}$$

where the terms $D_{ik} = D_{ik}(x,y)$; $i,k = [1,2,6]$ are a function of the $x$–$y$ coordinates. Thus, Eqs. (5) and (7) represent coupled fourth order nonlinear elliptic partial differential equations in terms of stress function $\Omega$ and transverse deflection $w$ with variable coefficients for postbuckling analysis of VAT composite plates. The stress boundary conditions expressed in terms of $\Omega$ and its derivatives were applied along the boundary. For axial compression loading, the boundary conditions are given by,

$$\begin{aligned} &\frac{\partial^2\Omega}{\partial y^2}|_{x=0,a} = \sigma_x(y), \quad \frac{\partial^2\Omega}{\partial x^2}|_{y=0,b} = 0, \quad \frac{\partial^2\Omega}{\partial x \partial y}|_{x=0,a;y=0,b} = 0, \\ &\Omega_{x=0,y=b} = \frac{\partial\Omega}{\partial x}|_{x=0,y=b} = \frac{\partial\Omega}{\partial y}|_{x=0,y=b} = 0. \end{aligned} \tag{8}$$