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Short communication

Some exact solutions for rotating flows of a generalized Burgers' fluid in cylindrical domains

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ABSTRACT

The velocity field and the adequate shear stress corresponding to the flow of a generalized Burgers' fluid model, between two infinite co-axial cylinders, are determined by means of Laplace and finite Hankel transforms. The motion is due to the inner cylinder that applies a time dependent torsional shear to the fluid. The solutions that have been obtained, presented in series form in terms of usual Bessel functions $J_1(\bullet)$, $J_2(\bullet)$, $Y_1(\bullet)$ and $Y_2(\bullet)$, satisfy all imposed initial and boundary conditions. Moreover, the corresponding solutions for Burgers', Oldroyd-B, Maxwell, second grade, Newtonian fluids and large-time transient solutions for generalized Burgers' fluid are also obtained as special cases of the present general solutions. The effect of various parameters on large-time and transient solutions of generalized Burgers' fluid is also discussed. Furthermore, for small values of the material parameters, λ_2 and λ_4 or λ_1, λ_2, λ_3 and λ_4, the general solutions corresponding to generalized Burgers' fluids are going to those for Oldroyd-B and Newtonian fluids, respectively. Finally, the influence of the pertinent parameters on the fluid motion, as well as a comparison between models, is shown by graphical illustrations.

1. Introduction

Non-Newtonian fluids (such as polymer solutions, fresh concrete, sludge, mud flows, food pastes, blood, paints, certain oils and greases, cosmetic fluids, etc.) in many areas of life have generated increased interest in their study. Typical non-Newtonian characteristics include shear-thinning, shear-thickening, viscoelasticity, viscoplasticity (i.e. the exhibition of an apparent yield stress) and so on. Non-Newtonian fluids form a broad class of fluids in which the relation connecting the shear stress and the shear rate is non-linear and hence there is no universal constitutive model available which exhibits the characteristics of all non-Newtonian fluids. Moreover, due to the flow behavior of non-Newtonian fluids, the governing equations become more complex to handle as additional non-linear terms appear in the equations of motion. Some interesting recent studies regarding these fluids are presented in [1–9].

Many models are accorded to describe the rheological behavior of non-Newtonian fluids [10,11]. They are usually classified as fluids of differential, rate and integral type. Amongst the non-Newtonian fluids, the rate type fluids are those which take into account the elastic and memory effects. The simplest subclasses of rate type fluids are those of Maxwell and Oldroyd-B fluids. But these fluid models do not exhibit rheological properties of many real fluids such as asphalt in geomechanics and cheese in food products. Recently, a thermodynamic framework has been put into place to develop the one-dimensional rate type model known as Burgers' model [12] to the frame-indifferent three-dimensional form by Krishnan and Rajagopal [13]. This model has been successfully used to describe the motion of the earth's mantle. The Burgers' model is the preferred model to describe the response of asphalt and asphalt concrete [14]. This model is mostly used to model other geological structures, such as Olivine rocks [15] and the propagation of seismic waves in the interior of the earth [16]. We here mention some of the studies [17–25] made by using Burgers' model.

The motion of a fluid in cylindrical domains has applications in the food industry, oil exploitation, chemistry and bio-engineering, etc., it being one of the most important problems of motion near translating or rotating bodies. The first exact solutions corresponding to motions

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of non-Newtonian fluids in cylindrical domains seem to be those of Ting [26] for second grade fluids, Srivastava [27] for Maxwell fluids and Waters and King [28] for Oldroyd-B fluids. In the meantime a lot of papers regarding such motions have been published. The most of them deal with motion problems in which the velocity field is given on the boundary (see [29] and the references therein, for instance). Unfortunately, there exists too little work for motions of non-Newtonian fluids due to a circular cylinder that applies a shear stress to the fluid [30–37].

The aim of the present article is to extend the results of Bandelli and Rajagopal [30, Sect. 4] to rate type fluids, namely to Oldroyd-B, Burgers' and generalized Burgers' fluids. The general solutions for Oldroyd-B fluids can be immediately specialized to give the similar solutions for Maxwell, second grade and Newtonian fluids, while the solutions for Burgers' fluids can be obtained as limiting cases of those for generalized Burgers' fluids. These solutions satisfy both the governing equations and all imposed initial and boundary conditions. The large-time and transient solutions for generalized Burgers' fluid, are also obtained and the effect of various parameters on transient and large-time solutions are discussed. Furthermore, for small values of the material parameters, λ_2 and λ_4 or λ_1, λ_2, λ_3 and λ_4, the general solutions corresponding to generalized Burgers' fluids are going to those for Oldroyd-B and Newtonian fluids, respectively. Finally, in order to reveal some relevant physical aspects of the obtained results, the diagrams of the velocity and the shear stress are depicted against r for different values of t and of the pertinent parameters. A comparison between different models is also presented.

2. The differential equations governing the flow

The equations governing the flow of an incompressible fluid include the continuity equation and the momentum equation. In the absence of body forces, they are

$$\nabla \cdot \mathbf{V} = 0, \tag{1}$$

$$\nabla \cdot \mathbf{T} = \rho \frac{\partial \mathbf{V}}{\partial t} + \rho(\mathbf{V} \cdot \nabla)\mathbf{V}, \tag{2}$$

where ρ and $\mathbf{V}$ are respectively the fluid density and velocity field and ∇ represents the gradient operator.

The Cauchy stress tensor $\mathbf{T}$ for an incompressible generalized Burgers' fluid is characterized by the following constitutive equations [21–25]

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} + \lambda_1 \frac{\delta \mathbf{S}}{\delta t} + \lambda_2 \frac{\delta^2 \mathbf{S}}{\delta t^2} = \mu \left[\mathbf{A} + \lambda_3 \frac{\delta \mathbf{A}}{\delta t} + \lambda_4 \frac{\delta^2 \mathbf{A}}{\delta t^2} \right], \tag{3}$$

where $-p\mathbf{I}$ denotes the indeterminate spherical stress, $\mathbf{S}$ is the extra-stress tensor, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ is the first Rivlin–Ericksen tensor ($\mathbf{L}$ being the velocity gradient), μ is the dynamic viscosity, λ_1 and $\lambda_3(<\lambda_1)$ are relaxation and retardation times, λ_2 and λ_4 are the new material parameters of the generalized Burgers' fluid (having the dimension of t^2), and $\delta/\delta t$ denotes the upper convected derivative defined by [21–25]

$$\frac{\delta \mathbf{S}}{\delta t} = \frac{d\mathbf{S}}{dt} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T, \qquad \frac{\delta^2 \mathbf{S}}{\delta t^2} = \frac{\delta}{\delta t}\left(\frac{\delta \mathbf{S}}{\delta t}\right). \tag{4}$$

Into above relation, d/dt is the usual material time derivative.

This model includes as special cases the Burgers' model (for $\lambda_4 = 0$), Oldroyd-B model (for $\lambda_2 = \lambda_4 = 0$), Maxwell model (for $\lambda_2 = \lambda_3 = \lambda_4 = 0$) and the Newtonian fluid model when $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$. In some special flows, like those to be here considered, the governing equations corresponding to generalized Burgers' fluids resemble those for second grade fluids.

For the problem under consideration we assume a velocity field $\mathbf{V}$ and an extra-stress tensor $\mathbf{S}$ of the form

$$\mathbf{V} = \mathbf{V}(r,t) = w(r,t)\mathbf{e}_\theta, \qquad \mathbf{S} = \mathbf{S}(r,t), \tag{5}$$

where $\mathbf{e}_\theta$ is the unit vector in the θ-direction of a cylindrical coordinate system r, θ, z. For these flows the constraint of incompressibility is automatically satisfied. If the fluid is at rest up to the moment $t = 0$, then

$$\mathbf{V}(r,0) = \mathbf{0}, \qquad \mathbf{S}(r,0) = \frac{\partial \mathbf{S}(r,0)}{\partial t} = \mathbf{0}, \tag{6}$$

and Eqs. (3) and (5) imply $S_{rr} = S_{\theta z} = S_{zr} = S_{zz} = 0$.

The balance of the linear momentum, in the absence of a pressure gradient in the axial direction ($\partial_\theta p = 0$ due to the rotational symmetry [33]), and the constitutive equation corresponding to generalized Burgers' fluid lead to the relevant partial differential equations [19,25]

$$\left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right) \tau(r,t) = \mu \left(1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2}\right) \left(\frac{\partial}{\partial r} - \frac{1}{r}\right) w(r,t), \tag{7}$$

$$\rho \frac{\partial w(r,t)}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{2}{r}\right) \tau(r,t), \tag{8}$$

where $\tau(r,t) = S_{r\theta}(r,t)$ is the shear stress which is different of zero and μ is the coefficient of viscosity of the fluid. Eliminating τ between Eqs. (7) and (8), we attain to the governing equation

$$\left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right) \frac{\partial w(r,t)}{\partial t} = \nu \left(1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2}\right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) w(r,t), \tag{9}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid.