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Influence of parameter mismatch on the convergence of the band structures by using the *Fourier* expansion method

Zhibao Cheng, Zhifei Shi *

School of Civil Engineering, Beijing Jiaotong University, Beijing 100044, China

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ABSTRACT

In the present paper, the convergence of the *Fourier* expansion method in calculating the band structures of elastic waves propagating in periodic composites is discussed. First, the convergence of the partial sum of a periodic function with discontinuous points is investigated by introducing three parameters to describe the overall error of a *Fourier* series and the *Gibbs* oscillation region. Second, the convergence of a product of two discontinuous functions is discussed based on two approximation methods, i.e., *Laurent* theory and the inverse formula, respectively. Finally, as an application, elastic waves propagating in one-dimensional layered periodic composite are studied. The band structures of the layered composites are obtained and some major influencing factors are discussed. Attentions are concentrated on the influences of the mismatch of mass density, shear modulus and filling ratio on the convergence of dispersion curves.

1. Introduction

Composite materials, especially periodic composite materials, are widely used in many engineering fields. The classical research topic on the linear framework solution of periodic reinforced particulate composite is well described in the earlier publications [1,2]. Over the past several decades, the static properties of periodic composites have been extensively investigated based on homogenization method. By introducing the numerical homogenization technique and periodic boundary conditions, Kari et al. [3] evaluated the effective material properties of spherical particle reinforced composites for different volume fractions. Combining the second-order homogenization method and a fast *Fourier* transform algorithm, Idiart et al. [4] studied the macroscopic behavior and local field distributions in a special class of two-dimensional periodic composites with viscoplastic phases. Using a modified version of the asymptotic expansion homogenization method, Chatzigeorgiou et al. [5,6] studied the effective thermomechanical properties of composites with cylindrical periodicity and generalized periodicity in the microstructure. Combining self-consistent and numerical methods, Kanaun et al. [7,8] studied the static elastic and elasto-plastic fields of 3D-composite materials with periodic and random microstructures, the effective properties of these materials were also presented. Andrianov et al. [9] provided a detailed investigation on the application of asymptotic homogenization techniques to the analysis of viscoelastic-matrix fibrous composites with periodic cell reinforcement and their effective properties. Using the numerical response function method, Kamiński [10] determined the sensitivity coefficients of the elastic effective properties of periodic fiber-reinforced composites.

Dynamic properties of periodic composites have also got much attention by many researches. It is realized that the inherent property of periodic composites, called band of frequency gaps, may help to design new composite materials for a large variety of engineering applications, such as a vibrationless environment for high-precision mechanical systems, acoustic filters and noise control devices, and ultrasonic transducers. Andrianov et al. [11,12] conducted a comparison of different methods determining band of frequency gaps in 1D and 2D periodic heterogeneous media. In their investigations, the well-known dispersion equation for the 1D case is studied via asymptotic approach. In addition, the geometrical nonlinearity described by the *Cauchy–Green* strain tensor and the physical nonlinearity described by elastic potential were taken into account. Their investigation shows that the *Fourier* expansion method works well for relatively small inclusions. Vivar-Pérez et al. [13] proposed a dispersive asymptotic method and studied the wave propagation in a kind of periodic composites. Assuming both the spherical solid inclusions and the matrix to be thermo-elastic and using the self-consistent method, Valdiviezo-Mijangos et al. [14] studied the dispersion and attenuation of shear waves propagating in multi-size particulate composites. Using the homogenization method, Carta and Brun [15] studied the elastic waves propagating in a periodic laminate. Their investigation on the long-wave asymptotic approximation of the model shows that the proposed approach agrees with the exact solution in a wide

* Corresponding author. Tel.: +86 10 51688367.
E-mail addresses: zfshi178@bjtu.edu.cn, zfshi178@sohu.com (Z. Shi).

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range of elastic impedance contrasts. Nemat-Nasser and Srivastava [16–18] studied the overall dynamic properties of both layered elastic composites and three-dimensional periodic elastic composites. Their investigations on three-dimensional periodic elastic composites show that the overall compliance (stiffness) tensor is *Hermitian*, regardless of whether the corresponding unit cell is geometrically or materially symmetric [16]; the frequency-dependent effective properties may become negative for cases where there exists a possibility of local resonance below the length-scale of the wavelength [17]. Considering some typical continuous and discrete models of structure, Movchan and Slepyan [19] developed a fully analytical framework for analysis of localized vibrations within certain classes of periodic structures, in which the concept of negative mass and/or negative stiffness is well-explained. Using differential quadrature method, Xiang and Shi [20] obtained the flexural vibration band gaps in periodic beams. In addition, Shi et al. [21,22] introduced the concept of band of frequency gaps to the field of civil engineering and proposed a new seismic isolation method by using a periodic foundation. By using Bloch–Floquet theory, Brun et al. [23] addressed a mathematical model describing the dynamic response of an elongated bridge supported by elastic pillars.

Several methods have been successfully used to analyze the dispersion structure of periodic composites, such as the plane wave expansion (PWE) method, the multiple-scattering theory (MST), the finite difference time domain method (FDTD), the finite element method (FEM), the lumped mass method (LM) and the transfer matrix method (TM). A brief summary of these methods is given in [20]. Specially, PWE method has been extensively used for calculating band structures or field distribution because of its convenience, in which the wave equations are solved in the *Fourier* space [24]. This method exhibits flexibility in handling different types of dispersion relationship for periodic structures, like bulk wave in 2D isotropic elastic composite [25,26], Lamb elastic waves propagating in both thick and thin plates [27,28], surface wave in two dimensional isotropic elastic composite [29] and so on.

However, the PWE method approximates unknown fields in a heterogeneous medium by some infinite *Fourier* series expansions, which may exhibit convergence difficulties for densely packed high-contrast composites [11,30–32]. Generally speaking, there are two different points on the reason of the slow convergence for PWE method. One believes that the slow convergence is due to the slow convergence of the *Fourier* series [32]. Another believes that the slow convergence is not due to the slow convergence of the *Fourier* series for the elastic coefficients (or the displacement fields) in the interfaces of different materials, but to the inappropriate formula used in the calculation [31]. Kuratsubo et al. [33,34] investigated the uniform convergence for the *Fourier* series and the *Fourier* inversion, and conducted a study on *Gibbs–Wilbraham* phenomenon and the arc length of *Fourier* series. Their results showed that the arc length of the graphs *Gamma* of the partial sum of the *Fourier* series of a piecewise function with jump discontinuities is asymptotically equal to the product of the sum of all jumps of the function and the *Lebesgue* constant. Adcock [35] detailed the *Gibbs* phenomenon and its removal for a class of orthogonal expansions consisting of eigenfunctions of univariate polyharmonic operators equipped with homogeneous *Neumann* boundary conditions and showed that this phenomenon closely resembles the classical *Fourier Gibbs* phenomenon at interior discontinuities. In addition, Adcock [36] also extended Eckhoff's method to the convergence acceleration of multivariate modified *Fourier* series. Through removing the *Gibbs* phenomenon by use of signal-filtering-based concepts and some properties of the *Fourier* series, Wangüemert-Pérez et al. [37] presented a simple strategy for accurately recovering discontinuous functions from their *Fourier* series coefficients. By considering a higher number of discretization points in the space domain, Ortega-Moñux et al. [38] presented a nonconventional variant of a full-vector fast-*Fourier*-transform-based mode solver to analyze arbitrarily shaped optical dielectric waveguides, and conducted an accurate analysis of arbitrarily shaped index-guiding photonic crystal fibers [39]. Based on the application of Sobolev spaces with variable order, Babuška and Osborn [40] derived the convergence results and error estimates of the eigen values of a second order differential equation, in which the second order equations are factored into a first order system and the Ritz-Galerkin method is applied to the system.

In the present paper, the convergence of *Fourier* expansion method in calculating the dispersion relationship of elastic waves propagating in periodic composite is studied. Two mathematical approximations are considered to find the reason of slow convergence: one is to approximate the discontinuous function by using its partial sum, and another is to approximate the *Fourier* coefficients of the product function ($h(x) = f(x)g(x)$) by using the coefficients of functions $f(x)$ and $g(x)$ as applying the convolution theorem. For the first mathematical approximation, *Gibbs* phenomenon at discontinuous interfaces is the main reason for slow convergence, which will be discussed in Section 2. For the second approximation, non-uniform approximation of the partial sum of a product function, especially when the product function is continuous, is the governing factor for slow convergence, which will be investigated in Section 3. To make it more understandable, several mathematical examples are given in the above two sections. In Section 4, an infinite layered periodic structure is used to illustrate the influences of the mathematical approximations. The accuracy among different methods is compared. Numerical investigations are conducted and attentions are focused on the influence of physical and geometrical parameters of composites on the convergence. Some useful results are presented in Section 5.

2. Expansion of a periodic function and *Gibbs* oscillation

Let periodic function $f(x)$ be an integrable function and satisfying *Dirichlet* conditions, then $f(x)$ can be expanded to a *Fourier* series $S[f]$. The partial sum $S_n(x)$ of *Fourier* series $S[f]$ converges to $f(x)$ at any continuous points or converges to $[f(x_j + 0) + f(x_j - 0)]/2$ at the first kind of discontinuous points x_j. However, it should be pointed out that *Gibbs* oscillation exists in the region near a discontinuous point, which has a significant effect on the convergence of *Fourier* series [35,41,42].

Periodic composite materials are made of one unit in a periodic manner and have been widely used in many engineering fields. In order to simplify the analysis, material parameters such as density and Young modulus of a periodic composite are often assumed to be periodic functions. Sometimes, the responses such as displacement and stress as well as strain of a periodic composite may also be periodic functions. Most of these functions have first kind of discontinuous points at interfaces. Therefore, *Gibbs* oscillation exists inevitably as using the partial sum of these functions in numerical calculation, which seriously deteriorates the convergence of *Fourier* expansion method. In the following analysis, the convergence of a *Fourier* series is studied and attentions are focused on the *Gibbs* oscillation.

2.1. Gibbs oscillation of a periodic function

Two periodic functions $f(x)$ and $g(x)$ as well as their partial sums $f_N(x)$ and $g_N(x)$ are introduced as follows:

$$f(x) = \begin{cases} S & |x| \leqslant \alpha, \\ 1 & \alpha < |x| \leqslant 1. \end{cases} \quad f(x) \approx f_N(x) = \sum_{n=-N}^{N} F_n e^{inx}; \tag{1a}$$