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Research Paper

Numerical solution of Richards' equation of water flow by generalized finite differences

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ABSTRACT

Richards equation is a degenerate elliptic parabolic nonlinear expression which models flow in unsaturated porous media. Due to its importance in engineering, a number of linearization schemes for approximating its solution have been proposed. Among the more efficient are combinations of Newtonian iterations for the spatial discretization using finite elements, and an implicit θ-method for the time integration. However, when the finite element formulation is used, numerical oscillations near the infiltration front are presented. To overcome this problem, this paper presents a novel generalized finite differences scheme and an adaptive step size Crank-Nicolson method, which can be applied for solving Richards' equation on nonrectangular structured grids. The proposed method is tested on an illustrative numerical example on a road embankment and the results are compared with a finite element method solution.

1. Introduction

The construction of a road involves the placement of porous natural materials to form embankments and pavements. Once a structure has been built, it is subject to changing weather conditions; one of those is water infiltration. Changes in moisture content in the soil can decrease the resistance and cause collapse deformations [2,5,25]. For this reason, we are interested in modeling water infiltration. Nowadays, in the design of new pavement, several methodologies that consider the effect of water content on the mechanical properties of the soils are being used [1].

The hydrodynamics of the infiltration process is described by Richards' equation [27], a nonlinear equation which requires a thorough numerical treatment to approximate its solutions. In one dimensional problems, the finite difference method is one of the most used methodologies to solve Richards' equation [17,20,26]. In the case of two-dimensional domains grids with rectangular cells are often used; in slopes, boundaries are approximated using active grid cells, and this produces stepped grids, for which standard finite differences are no longer useful [12,17,15], but they can be treated using the Finite Element Method (FEM), which has been tested in several applications [9,14,16,24].

In both cases, anyway, a crucial aspect is the time step adaptation. The discretization of some constitutive models leads to numerical convergence difficulties that arise from discontinuities in the time derivatives. To solve this problem, several authors have studied the effect of different time integrations in the solution. Celia [3] addressed the oscillatory solutions produced by finite element discretizations and proposes the use of mass conserving approximations based on the mixed form of Richards' equation to produce robust numerical solutions. The same author also points out that time derivative is fundamental to avoid oscillations near the infiltration front, and suggests the use of second order Backward Differentiation Formulas. Miller [20] considered two discretizations: a backward difference with better results than those of Celia [3], as well as a numerical technique called DASK, which combines backward Euler with a Newtonian iteration in Krylov spaces, to integrate a high order algebraic method of lines. The latter provides a robust and accurate approximation of variably saturated flow. Kavetski [13] used an automatic adaptive backward Euler time stepping with truncation control, and an efficient time step selector, to control the temporal accuracy of the integration. He presented several tests that show the effectiveness of the automatic optimization of the time step size.

The overall of the previous analysis shows that the finite element method requires a lumping procedure for the mass matrix in the time integration. On the contrary, the usual finite differences always produced diagonal matrices in the time derivative, but they could only be applied on block-rectangular meshes. On the other hand, in the case of two-dimensional problems, the Generalized Finite Differences Method (GFDM) has recently been used to produce satisfactory approximations

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to solve differential equations on structured grids over irregular regions [7]. In this study, bearing in mind that in geotechnical engineering applications the domains often have irregular borders, the combinations of GFDM for the space derivatives, as well as an adaptive step size algorithm for the time derivative, is presented as a promising alternative for solving Richards' equation.

The paper is organized as follows: we address Richards' equation in Section 2. An iterative scheme based on generalized finite differences on structured grids, as well as the adaptive time stepping, is discussed in Section 3. A solution of a typical example of the infiltration of an embankment is presented in Section 4. Finally, the efficiency of the proposed approach is discussed in Section 5.

2. Richards' equation

In the description of flow in saturated and unsaturated soils, the hydraulic head h is used as a measure of the energy of water in the soil. The head is expressed as

$$h = y + \frac{u_w}{\gamma_w} \tag{1}$$

where

y is the gravitational head,
u_w is the pore water pressure, and
γ_w is unit weight of water.

From Eq. (1), pore water pressure is given by

$$u_w = (h - y)\gamma_w. \tag{2}$$

To describe water flow through unsaturated soil it is necessary to define the matric suction,

$$s = u_a - u_w \tag{3}$$

where u_a is the pore air pressure in the pores of the soil.

Suction is considered as a state variable which can describe the physical behavior of unsaturated soils [2,10,27,29]. Some hydraulic properties of soil, as volumetric water content and permeability, are functions of suction.

The classical description of flow rate for saturated soils is given by Darcy's law, which is valid for unsaturated soils. This law states that the water flow rate through a soil is directly proportional to the hydraulic head gradient, and it is written for a one-dimensional flux as

$$v_\xi = -k_\xi(s)\frac{dh}{d\xi} \tag{4}$$

where

v_ξ is the water flow rate along the ξ axis,
$k_\xi(s)$ is the water permeability coefficient, and
$\frac{dh}{d\xi}$ is the hydraulic head gradient along the ξ direction.

The Richards' equation is derived from mass conservation and Darcy's law, and is given as

$$\frac{\partial}{\partial x}\left(k_x\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y\frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z\frac{\partial h}{\partial z}\right) = m_w\gamma_w\frac{\partial h}{\partial t} \tag{5}$$

where
$m_w = \frac{\partial\theta_w}{\partial s}$ is the water storage modulus, and
θ_w is the volumetric water content.

In addition, to complete the flow description, one must consider a retention curve. For instance, the one given by Van Genuchten equation [29]

$$\theta_n(s) = \frac{1}{((\alpha s)^n + 1)^m}, \tag{6}$$

where

$\theta_n = \frac{\theta_w - \theta_r}{\theta_s - \theta_r}$ is the normalized volumetric water content,
θ_r is the residual volumetric water content,
θ_s is the saturated volumetric water content,
α is a fitting parameter related to air-entry pressure (1/kPa) and
n,m are curve fitting parameters, and $m = 1-\frac{1}{n}$.

Alternatively, the Fredlund and Xing equation can be used [11]

$$\theta_n(s) = \frac{1}{\left(\ln\left(e + \left(\frac{s}{a_f}\right)^{n_f}\right)\right)^{m_f}} \tag{7}$$

where

a_f is a fitting parameter related to air-entry pressure (kPa),
n_f,m_f are fitting parameters.

Water permeability is a function of matric suction and volumetric water content. A common expression for permeability is Van Genuchten-Mualem Model [23,29], which is given by

$$k(\theta_n) = k_s\sqrt{\theta_n}\,(1-(1-\theta_n^{1/m})^m)^2, \tag{8}$$

where k_s is the saturated conductivity of soil, and m was described in Eq. (6).

3. Finite difference approximation

Once Richards' equation has been posed, the next step in the modeling process is to approximate its solution in terms of pore water pressure on a xz cross section of an embankment. The latter is assumed to be highly elongated along the y axis. Hence, the dimensionality of the problem eliminates the y direction and this yields the expression

$$\frac{k_x\partial^2 u_w}{\partial x^2} + \frac{k_z\partial^2 u_w}{\partial z^2} + \frac{\partial k_x}{\partial x}\frac{\partial u_w}{\partial x} + \frac{\partial k_z}{\partial z}\frac{\partial u_w}{\partial z} + \gamma_w\frac{\partial k_z}{\partial z} = \gamma_w m_w\frac{\partial u_w}{\partial t}. \tag{9}$$

A common discretization procedure to approximate the solutions of differential equations is the use of finite difference schemes, due to the fact that they can be calculated easily in block-rectangular regions. Linearity and nonlinearity of the differential operator give rise to linear and nonlinear systems of equations, respectively.

Despite the basic idea of Taylor's Theorem is quite simple, its use leads to more complicated schemes on nonrectangular regions. Although in all the cases the main deduction requires mainly only calculus in several variables. There are some interesting approaches due to Liszka [19] for regular non-rectangular geometries and to Benito [21,22,28] for slightly perturbed clouds of nodes. Some schemes for the classical heat equation in irregular regions were presented in [7].

In this paper, we address the use of a general difference scheme that can be calculated by considering a finite set of nodes $p_0 = (x_0,z_0), p_1 = (x_1,z_1), p_2 = (x_2,z_2),\ldots,p_k = (x_k,z_k)$, for which it is required to find coefficients $\Gamma_0,\Gamma_1,\ldots,\Gamma_k$ in such that consistency is fulfilled. Thus, the local truncation error to approximate the second order linear operator

$$\begin{aligned}L(u_w,A(x,z),B(x,z),C(x,z),D(x,z),E(x,z)) &= A(x,z)\frac{d^2u_w}{dx^2} + B(x,z)\frac{d^2u_w}{dz^2}\\ &\quad + C(x,z)\frac{du_w}{dx} + D(x,z)\frac{du_w}{dz} + E(x,z)u_w\end{aligned} \tag{10}$$

at the grid point p_0 by means of the combination