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Research Paper

A Petrov-Galerkin finite element method for 2D transient and steady state highly advective flows in porous media

W. Cui*, K.A. Gawecka, D.M. Potts, D.M.G. Taborda, L. Zdravković

Department of Civil and Environmental Engineering, Imperial College London, London SW7 2AZ, UK

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ABSTRACT

A new Petrov-Galerkin finite element method for two-dimensional (2D) highly advective flows in porous media, which removes numerical oscillations and retains its precision compared to the conventional Galerkin finite element method, is presented. A new continuous weighting function for quadratic elements is proposed. Moreover, a numerical scheme is developed to ensure the weighting factors are accurately determined for 2D non-uniform flows and 2D distorted elements. Finally, a series of numerical examples are performed to demonstrate the capability of the approach. Comparison against existing methods in the simulation of a benchmark problem further verifies the robustness of the proposed method.

1. Introduction

Many engineering problems involve heat or solute transport by means of both diffusion (or conduction) and advection. In those cases where the latter is dominant, the flow is referred to as highly advective. Examples where this phenomenon applies are ground source energy systems (GSES) which utilise the thermal energy beneath the earth's surface, such as open-loop GSES.

In recent years, considerable effort has been placed into researching the GSES systems due to their attractiveness as renewable energy sources. In many cases, numerical methods have been employed to investigate their behaviour and performance. However, finite element modelling of highly advective flows can often be problematic as the extensively used Galerkin finite element method (GFEM) may result in spatial oscillations of the nodal solution. Several studies (e.g. [1–3]) have shown that the oscillations become more significant with increasing Péclet number, which is a function of the properties of the porous medium governing diffusion and the velocity of the fluid, as well as the finite element size. Recently, Cui et al. [4] have also demonstrated that the type of the element and the boundary condition employed affect the behaviour and magnitude of the oscillations. Although the oscillations can be eliminated by refining the finite element mesh, in problems such as those involving open-loop GSES, this approach results in an extremely large number of elements, thus becoming computationally expensive.

The deficiencies of the GFEM have led to the development of various upwind finite element methods to model highly advective flows. The three main methods are ([5]): artificial diffusion, quadrature and Petrov-Galerkin (PG). Artificial (or balancing) diffusion involves the introduction of an extra term to the physical diffusion in the advection-diffusion equation which can then be solved using the standard GFEM. The main advantage of this approach is that it is easy to implement. However, it may result in a reduction of accuracy during the transient stage, as it alters the physics of the problem ([3]). The quadrature technique was developed by Hughes [6] who suggested moving the quadrature (or integration) points within the element for a more efficient upwind effect. Conversely, the basic principle behind the PG formulation is to modify the nodal weighting functions in order to weigh the contribution from the upstream element more heavily than that from the downstream one.

The first upwind formulation for steady state advection-diffusion problems was proposed by Christie et al. [7] who modified the weighting function for a one-dimensional (1D) linear element. Later, the same approach was extended to two-dimensional (2D) linear elements by Heinrich et al. [8], as well as 1D and 2D quadratic elements by Heinrich and Zienkiewicz [9]. Huyakorn [10] suggested using different expressions for the weighting functions for 1D and 2D linear elements. Upwinding of 1D cubic elements was explored by Christie and Mitchell [11]. Kelly et al. [12] extended the technique to simulate a steady state problem with a uniform inclined flow in a 2D mesh by adding an extra diffusion term in the direction of flow to the advection–diffusion equation. Later, Donea et al. [13] explored a different modification to the steady state governing equation by studying quadratic elements. It should be noted that modifying the weighting functions has the same effect as adding artificial diffusion, provided only one-dimensional steady state is considered. Hence, most of the abovementioned methods

* Corresponding author at: Department of Civil and Environmental Engineering, Imperial College London, London SW7 2AZ, UK.
E-mail address: w.cui11@imperial.ac.uk (W. Cui).

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produce exact solutions for steady state advection–diffusion problems.

The problem of transient advection–diffusion was found to be more challenging. The first solution to the transient problem for linear 1D and 2D elements was proposed by Huyakorn and Nilkuha [14] and Ramakrishnan [15]. While Huyakorn and Nilkuha [14] used the same continuous weighting functions for linear elements as Huyakorn [10], the time derivative term was weighted using the standard Galerkin weighting functions which resulted in oscillating and over-diffused solutions. Other researchers attempted to overcome the problem by developing higher order weighting functions. Cubic expressions were adopted by Ramakrishnan [15], who observed a reduction in accuracy despite applying the modified weighting functions to all terms in the advection–diffusion equation. Later, Dick [16] proposed weighting functions which are a combination of continuous cubic and quadratic expressions for 1D and 2D linear elements, whereas Westerink and Shea [17] extended this approach to quartic weighting functions for 1D quadratic elements. Cardle [18] applied different weighting functions to the time derivative and the spatial derivative in the 1D advection–diffusion equation. Another approach to the transient problem was suggested by Yu and Heinrich [19,20] who developed continuous weighting functions which include time dependency (the bilinear time–space shape functions). The same method with different expressions for the time–space shape functions was adopted by Al-Khoury and Bonnier [21].

Brooks and Hughes [5] developed the streamline upwind/Petrov-Galerkin (SU/PG) method where discontinuous weighting functions, instead of the continuous functions mentioned above, were used. In this approach, multi-dimensional flows were also considered. However, the weighting functions as well as the adopted weighting factors for multi-dimensional cases were formulated provisionally, and Brooks and Hughes [5] suggested further research should be carried out to obtain a more rigorous approach for multi-dimensional cases. Based on the SU/PG approach, further development and modification to this approach were proposed in Hughes et al. [22] and Tezduyar and Park [23] (Shock Capturing (SC) method), Hughes et al. [24] (Petrov-Galerkin Least Square (PGSL) method), and Tezduyar and Ganjoo [25] and Codina et al. [26] (time–space weighting functions). Compared to the original SU/PG method, improved results were observed when multi-dimensional highly advective flows were simulated. However, for the above-mentioned methods, only results of problems involving flow through a mesh with regular (e.g. square or rectangular) elements were shown, while the performance in problems using irregular finite element meshes has not been demonstrated in the literature.

It is evident that some of the methods mentioned above (both in one- and multi-dimensional forms) were observed to produce over-diffused solutions. This reduction in accuracy was automatically assigned by some researchers in this field to all upwind techniques (e.g. [27]). While this is true for the artificial diffusion approach, Brooks and Hughes [5] argued that the Petrov-Galerkin finite element method (PGFEM) does not experience this problem, provided appropriate modified weighting functions are employed.

In the current study, the PGFEM with continuous weighting functions is adopted and further developed. While examining the continuous weighting functions proposed in the literature, the authors concluded that the use of the more complex higher order expressions (e.g. [15,17]) is not necessary, and that the time–space shape functions (e.g. [19,20]) are not compatible with the time marching scheme employed in the current paper. Additionally, it was also found that the PG weighting formulation for 2D 8-noded quadratic elements proposed by Heinrich and Zienkiewicz [9] cannot eliminate oscillations in either steady state or transient analyses. In fact, in none of the above-mentioned studies using continuous weighting functions (e.g. [8–11]) was the PGFEM successfully applied to both transient and steady state problems.

As a result, a consistent PGFEM with continuous weighting functions for solving the time-dependent advection–diffusion equation is proposed here, using both linear and quadratic elements. This scheme adopts the weighting functions of Huyakorn [10] and Heinrich and Zienkiewicz [9] for 1D linear and 1D quadratic elements, respectively, while the approach of Heinrich et al. [8] and Huyakorn [10] is adopted for weighting functions of 2D linear elements. However, a new formulation for 2D quadratic elements is developed and presented in this paper as the continuous formulation proposed by Heinrich and Zienkiewicz [9] was found to be inadequate. Furthermore, the modified weighting functions are applied to all terms in the time-dependent advection-diffusion equation which was not the case in some of the existing approaches (e.g. [14]). Additionally, a new numerical scheme for determining the Péclet numbers for 2D elements is also proposed, to ensure that the method retains its accuracy in complex cases, such as those involving 2D non-uniform flows (where both the flow direction and velocity vary spatially) and those where 2D distorted elements may be present. It should be noted that none of the PGFEMs available in the literature has been shown to perform well for these types of problems.

For the purpose of demonstrating its capabilities, the newly proposed PGFEM in this paper has been implemented into the finite element software ICFEP ([28]), which is the authors' bespoke computational platform and therefore allows access to source code, which is crucial for this type of development. A series of numerical studies are then performed, demonstrating the accuracy and stability of the new formulation for both transient and steady-state cases at different levels of complexity. Comparisons were also made between the proposed method and other methods using discontinuous weighting functions, such as the SU/PG and the SC methods, in the simulation of a multi-dimensional case widely documented in the literature (e.g. [22,23]). Finally, the paper demonstrates the excellent performance of the proposed PGFEM in simulating challenging multi-dimensional flow problems involving distorted elements where both flow velocity and flow direction vary across an element, which have not been shown anywhere in the available literature.

2. Fundamentals of FE modelling of convective flows in porous media

2.1. Coupled thermo-hydraulic formulation

Numerical modelling of convective flows in porous media requires the simulation of pore fluid flow coupled with heat transfer through both diffusion and advection, which is generally referred to as a coupled thermo-hydraulic (TH) problem. For incompressible pore fluid flow in a fully saturated porous medium, the continuity equation must be satisfied, which can be expressed as:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} - Q^f = \frac{\partial \varepsilon_v}{\partial t} \tag{1}$$

where v_x, v_y and v_z are the components of the velocity of the pore fluid in the x, y and z directions, respectively, ε_v is the volumetric strain due to stress changes, Q^f represents any pore fluid sources and/or sinks, and t is time. The seepage velocity, $\{v_f\}^T = \{v_x \quad v_y \quad v_z\}^T$, is assumed to be governed by Darcy's law, which can be written as:

$$\{v_f\} = -[k_f]\{\nabla h\} \tag{2}$$

where $[k_f]$ is the permeability matrix and ∇h is the gradient of the hydraulic head. In a coupled thermo-hydraulic problem, if it is assumed that: (a) the effects of temperature gradients and spatial variations in fluid density (e.g. buoyancy-driven flows) on pore fluid flow through a fully saturated porous medium are negligible and (b) the solid phase is rigid, Eq. (1) reduces to the equation of seepage, which can be expressed as:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = Q^f \tag{3}$$

Assuming an instantaneous temperature equilibration between the fluid