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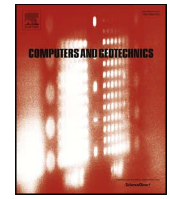

Research Paper

Minimum time-step size in transient finite element analysis of coupled poromechanical problems

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ABSTRACT

Finite element solution of poromechanical problems often exhibits oscillating pore pressures, if the time step is small relative to the spatial grid size. Here the nonphysical oscillations in the pore pressure are investigated and a novel analytical approach is presented, which satisfies the non-oscillatory criteria for compressible porous medium modeling. With the consistent and the lumped finite element schemes considered, the time-step constraints for three different types of one-dimensional elements are derived in terms of the mesh size and the material properties of both mechanics and seepage. Numerical examples are simulated to illustrate the obtained theoretical results.

1. Introduction

The theory of poromechanics, originally developed by Biot [1,2] and later generalized by many others, describes the interaction (or coupling) between the deformation and the fluid flow in a fluid-saturated porous medium. Biot's model is still widely used today in a great variety of fields, ranging from geomechanics and petroleum engineering ever since its establishment, to biomechanics [3] or even food processing [4]. Some examples of applications in engineering fields include petroleum production, nuclear waste disposal, carbon sequestration, soil consolidation, slope stability and hydraulic fracturing, and so on.

Although some analytical solutions have been derived for a few linear poroelastic problems [5], numerical simulations (e.g. finite element method [6]) seem to be the only way to obtain quantitative results for real applications. When the finite element method is adopted in transient analysis, the differential equation describing the problem is first integrated by a finite element discretization to approximate the numerical solutions in space. Then a time marching scheme, such as the well-known θ-method, is employed to approximate the numerical solutions over a time interval.

It is generally believed that decreasing the size of the adopted time step improves the accuracy of the numerical solutions to transient problems. However, approximations of the coupled poromechanical equations by standard finite element methods often exhibit strong nonphysical oscillations in the pore fluid pressure when time-step size is very small [7–9]. On the other hand, the oscillations may disappear on very fine grids, when some stability restrictions between the time and space discretization parameters are fulfilled [10]. For example, this is the case when standard linear finite elements or standard quadratic finite elements are used to approximate both displacement and pressure unknowns.

A long time back, Sandhu et al. [7] had observed the phenomenon that standard finite elements did not yield satisfactory solutions when a very small time-step size was used immediately after application of load in a consolidation analysis. Based on the analysis of the correlation among the time-step size, element length and physical parameters, Vermeer and Verruijt [10] proposed a minimum time-step size for the one-dimensional consolidation problem of porous media saturated with an incompressible fluid. The derived stability condition is based on the observation that the excess pore pressure due to an instantaneous load applied on a draining porous column cannot exceed the load itself. These authors used the elements with linear shape functions of pore pressures and suggested the same expression with a different multiplier for elements where pore pressures vary quadratically. They concluded that, to achieve stability, one had to refine the mesh until the minimum time step constraint was satisfied. From the stability analysis for a staggered solution, Turska and Schrefler [11,12] also found a lower limit for the time step in the cases of linear and nonlinear consolidations. Ferranato et al. [13] provided an empirical relation for a lower bound critical time-step size, below which ill-conditioning might suddenly occur. The minimum time-step size they defined is similar to that of Vermeer and Verruijt's criterion [10] and it is larger for soft and low permeable porous media discretized on coarser grids.

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In order to eliminate the nonphysical oscillations in pore fluid pressure, the approximation spaces satisfying an appropriate inf-sup condition [14] are commonly introduced. Such discretizations have been theoretically investigated by Murad et al. [15] However, an inf-sup stable pair of spaces like the Taylor-Hood elements [16], which approximates the displacement by continuous piecewise quadratic functions and the pressure by continuous piecewise linear functions, does not necessarily provide oscillation-free solutions [17].

Favino et al. [18] extended a strategy in one-dimensional case and derived the minimum value of the time-step size for the two-dimensional poromechanical problems by employing standard linear and Taylor-Hood square elements. Differently from the one-dimensional case, in the two-dimensional case the discrete maximum principle argument holds true only if the shear modulus of the solid skeleton is much less than the bulk modulus. More recently, applying the discrete maximum principle (DMP) [19] and the monotonicity restriction [20] as used in the heat conduction analysis, Cui et al. [21] had obtained time-step constraints in one-dimensional cases for linear, quadratic and Taylor-Hood elements, respectively, in coupled consolidation analysis and provided some suggestions to handle the numerical oscillation issues.

Although the time-step constraints for the finite element analysis of poromechanical problems have been established to a certain extent, most of the work is applicable to incompressible porous media only. The more general problems of fluid-saturated poroelastic media with compressible constituents have not been well addressed. On the other hand, the analysis of transient heat conduction shows that the lower bound of time-step size may disappear when the mass lumping techniques are employed in the transient context [22]. For the coupled poromechanical problems, however, the influence of mass lumping on the numerical stability remains unclear.

The objective of this paper is two-folded, i.e., to explore the mechanisms behind the nonphysical oscillations in the finite element solution of poromechanical problems, and to derive the minimum time-step sizes for the onset of spatial oscillations of pore fluid pressure. To these ends, both consistent and lumped Galerkin finite element schemes and different combinations of displacement and pore pressure shape functions are considered for one-dimensional problems. Although one-dimensional solutions are less applicable to practical scenarios, further research on them is still required, especially for higher-order elements in coupled analyses. Also, thorough understanding of the analytical approach for one-dimensional cases is necessary for the investigation of multidimensional problems. The latter was found to be much more difficult with complex geometries, various finite elements and changeable boundary conditions [23], whose critical time step might not be obtained analytically but had to be determined by trial and error [24], and therefore are not included here.

The paper is organized as follows. In Section 2, the mathematical model for the poromechanical problem is formulated and the finite element formulation with the generalized θ-method of time integration schemes is introduced. In Section 3, a criterion for numerical instability is established, and the existence of spatial oscillations at small time steps is shown. The non-oscillatory criteria are employed to derive minimum time-step sizes for the onset of the spatial oscillations, and to remove the instabilities from time-integration schemes. Some numerical examples are introduced to justify the theoretical arguments in Section 4.

2. Transient finite element formulation for poromechanical model

2.1. Governing equations

A detailed derivation of the coupled poromechanical equations can be found in Coussy's monograph [25]. Consider a porous medium composed of a solid matrix (s) saturated by a fluid (w). The porous medium has a porosity of ϕ, and its mass density, ρ, is given by $\rho = \phi\rho_w + (1-\phi)\rho_s$, where ρ_s and ρ_w are the intrinsic mass densities of the solid and the fluid, respectively. Let u be the displacement of the solid skeleton and p the pore fluid pressure (positive for compression). Assume that the skeletal deformation is infinitesimal and linear elastic and the fluid seepage velocity satisfies the Darcy's law. Under quasi-static conditions, the one-dimensional coupled formulations can then be written as

$$D\frac{\partial^2 u}{\partial x^2}-\alpha\frac{\partial p}{\partial x} + \rho g = 0, \tag{1}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) + \frac{1}{M}\frac{\partial p}{\partial t}-\frac{\partial}{\partial x}\left[\frac{k}{\mu}\left(\frac{\partial p}{\partial x}-\rho_w g\right)\right] = 0, \tag{2}$$

where x is the spatial coordinate, t the time, g the gravity acceleration, k the intrinsic permeability and μ the dynamic viscosity of the pore fluid. D is the constrained modulus, i.e., $D = K + 4G/3$, with K being the bulk modulus and G the shear modulus. α and M represent Biot's coefficient and Biot's modulus, respectively, which depend on the compressibility of the materials, namely,

$$\alpha = 1-\frac{K}{K_s}, \tag{3}$$

$$\frac{1}{M} = \frac{\alpha-\phi}{K_s} + \frac{\phi}{K_w}, \tag{4}$$

where K is the bulk modulus of the dry porous material; K_s and K_w are the bulk moduli of the solid phase and the fluid phase, respectively. It is worth mentioning that Biot's modulus is the inverse of the storage coefficient well known in groundwater applications [26], playing a significant role in the field of compressible fluid flow.

The saturated porous medium is assumed to occupy a space domain Ω with a boundary Γ. The initial conditions at $t = 0$ can be specified as

$$u = u_0, \quad p = p_0 \ (\text{in } \Omega). \tag{5}$$

The boundary conditions include

$$u = \hat{u} \ (\text{on } \Gamma_u), \quad p = \hat{p} \ (\text{on } \Gamma_p), \tag{6}$$

$$D\frac{\partial u}{\partial x} = \hat{t} \ (\text{on } \Gamma_\sigma), \quad \frac{k}{\mu}\frac{\partial p}{\partial x} = \hat{w} \ (\text{on } \Gamma_\omega), \tag{7}$$

where $\hat{u}$, $\hat{p}$, $\hat{t}$ and $\hat{w}$ are the specified displacement, fluid pressure, traction and flux on the boundaries Γ_u, Γ_p, Γ_σ and Γ_ω, respectively. Γ_u and Γ_σ are the non-overlapping portions of the boundary Γ, so as Γ_p and Γ_ω.

2.2. Spatial domain discretization

The standard Galerkin method is employed to develop the finite element formulation for the coupled poromechanical problem presented above. The skeletal displacement, u, and the pore fluid pressure, p, are chosen as the primary unknown variables. Let $\boldsymbol{N}_u$ and $\boldsymbol{N}_p$ be the shape function matrices for displacement and fluid pressure, respectively, and $\boldsymbol{U}$ and $\boldsymbol{P}$ are the corresponding vectors of unknowns. After discretizing the weak form of the linear partial differential Eqs. (1) and (2) in the spatial domain, one obtains the following set of fully coupled algebraic equations,

$$\begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{G}^T & \boldsymbol{C} \end{bmatrix}\begin{bmatrix} \dot{\boldsymbol{U}} \\ \dot{\boldsymbol{P}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{K} & \boldsymbol{G} \\ \boldsymbol{0} & \boldsymbol{H} \end{bmatrix}\begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{P} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_u \\ \boldsymbol{F}_p \end{bmatrix}, \tag{8}$$

where the matrices and vectors can be expressed as follows:

solid stiffness matrix $\boldsymbol{K} = -D\int_\Omega \left(\frac{\partial \boldsymbol{N}_u}{\partial x}\right)^T \frac{\partial \boldsymbol{N}_u}{\partial x}\mathrm{d}\Omega$,

coupling matrix $\boldsymbol{G} = \int_\Omega \left(\frac{\partial \boldsymbol{N}_u}{\partial x}\right)^T \alpha \boldsymbol{N}_p \mathrm{d}\Omega$,

compressibility matrix $\boldsymbol{C} = \frac{1}{M}\int_\Omega \boldsymbol{N}_p^T \boldsymbol{N}_p \mathrm{d}\Omega$,

permeability matrix $\boldsymbol{H} = \frac{k}{\mu}\int_\Omega \left(\frac{\partial \boldsymbol{N}_p}{\partial x}\right)^T \frac{\partial \boldsymbol{N}_p}{\partial x}\mathrm{d}\Omega$,