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Onset and dynamic shallow flow of a viscoplastic fluid on a plane slope

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ABSTRACT

The shallow flow of a viscoplastic fluid on a plane slope is investigated. The material constitutive law may include two plasticity (flow/no-flow) criteria: Von-Mises (Bingham fluid) and Drucker–Prager (Mohr–Coulomb). Coulomb frictional conditions on the bottom are included, which implies that the shear stresses are small and the extensional and in-plane shear stress becomes important. A stress analysis is used to deduce a Saint-Venant type asymptotic model for small thickness aspect ratio. The 2D (asymptotic) constitutive law, which relates the average plane stresses to the horizontal rate of deformation, is obtained from the initial (3D) viscoplastic model.

The "safety factor" (limit load) is introduced to model the link between the yield limit (material resistance) and the external forces distribution which could generate or not the shallow flow of the viscoplastic fluid. The DVDS method, developed in [I.R. Ionescu, E. Oudet, Discontinuous velocity domain splitting method in limit load analysis, Int. J. Solids Struct., doi:10.1016/j.ijsolstr.2010.02.012], is used to evaluate the safety factor and to find the onset of an avalanche flow.

A mixed finite element and finite volume strategy is developed. Specifically, the variational inequality for the velocity field is discretized using the finite element method while a finite volume method is adopted for the hyperbolic equation related to the thickness variable. To solve the velocity problem, a decomposition–coordination formulation coupled with the augmented lagrangian method, is adapted here for the asymptotic model. The finite volume method makes use of an upwind strategy in the choice of the flux.

Several boundary value problems, modeling shallow dense avalanches, for different visoplastic laws are selected to illustrate the predictive capabilities of the model.

1. Introduction

In last few years, a lot of efforts in geophysics and engineering have been devoted to the understanding of the physics of avalanche formation and to the shallow flow of soils, snow or other geomaterials over an inclined surface (see [1,35]). It has been recognized that the problem is 3D and that the behavior of the material is best represented by viscoplastic fluid type models. Since the numerical integration of the three-dimensional equations of viscoplastic fluids is very complex and poses many challenges, reduced 2D models, called also Saint-Venant models, are generally considered. Such models are able to capture the principal features of the flow: onset, dynamic propagation and arrest.

When the fluid is relatively shallow and spreads slowly, lubrication-style asymptotic approximations can be used to build reduced models for the spreading dynamics of viscoplastic fluids. For two-dimensional (sheet) flow lubrication models were introduced by Liu and Mei [25,26] and applied to problems of mud flow, while Balmforth et al. [3] considered the axisymmetric version of the problem to model the extrusion of lava domes. The lubrication model has been successfully extended to three dimensions in [2] and used thereafter in [4,5]. Other model, which considers the same adherence conditions on the bottom as the lubrication models, was recently obtained in [15].

When the movement is faster, shallow water theory for non-viscous flows may be used in conjunction with Coulomb frictional type boundary condition at the bottom. A depth integrated theory, obeying a Mohr–Coulomb type yield criterion, was introduced by Savage and Hutter [36,37] and developed thereafter by many authors (see for instance [23,41,28,29]). These models takes into account the frictional dissipation between the flowing layers parallel to the basal plane through an anisotropy factor which depends on the friction angles. The importance of this anisotropy factor is still an open question and an accurate derivation of the equations is still lacking (see e.g. [34]).

A large number of complex fluids exhibit an effective slip between the fluid and the wall. As it has been noted in [5], if slip becomes sufficiently severe, the flow can become relatively plug-like across the film thickness. Then, the shear stresses, which are dominant in the lubrication models become small, while the exten-

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sional and in-plane shear stress becomes important. This is the case when dealing with free liquid threads and films [32], ice shelves and streams [27] or snow avalanches. This situation demands a different theoretical model.

The main goal of this paper is to deduce Saint-Venant type (shallow flow) models for viscoplastic fluids/solids in frictional contact (no adherence) with a plane slope. Another objective is to study the capability of the obtained shallow model to capture the onset of the flow. More precisely, we seek to establish a criterion which relates the yield limit (material resistance) and the external forces distribution, to predict if the shallow flow occurs. A numerical scheme for solving the dynamic equations associated to the proposed asymptotic (shallow) model is the third objective of the paper.

Let outline the content of the paper. In Section 2 we state the three-dimensional problem. After a discussion on the adopted viscoplastic constitutive equation (Section 2.1) we present the geometry of the problem and the associated equations and boundary conditions. In Section 3, the asymptotic model is deduced. After the velocity and momentum scaling, we derive the shallow mechanical model, which links the in-plane averaged stresses to the in-plane rate of deformations. The shallow flow/no-flow condition and the shallow yield limit and viscosity are deduced. Thus, a complete set of governing equations and boundary conditions for the shallow problem are obtained. In the next section the "safety factor" (limit load) is introduced to study whenever the fluid/solid flows or not from a rest configuration. To compute the safety factor the DVDS method, developed in [21], is used. As an example, the onset of the shallow flow of a Bingham fluid with an uniform thickness is studied with the proposed approach. In Section 5 the sheet flow is analyzed. The comparison between the full (2D) in-plane flow and the asymptotic (1D) flow for the Bingham fluidis presented. We illustrate in this way the capabilities of the shallow model to reproduce the flow. In the same section we deduce analytical expressions of the safety factor. We conclude with some numerical illustrations of shallow flows: spreading a Drucker–Prager dome on a plane slope, the role of barriers in stopping a viscoplastic avalanche and the flow of a Bingham fluid from a reservoir. The proposed numerical scheme is presented in Appendix A.1. After an implicit time discretization the algorithm at each time step is developed. It makes use of a decomposition–coordination formulation coupled with the augmented lagrangian method in a finite element context, while an upwind flux strategy is adopted in a finite volume context.

2. The 3D-problem

2.1. The 3D viscoplastic model

Let us begin by describing the viscoplastic fluid model used in this paper. In contrast with a classical viscous fluid, which cannot sustain a shear stress at rest, the Cauchy stress tensor $\boldsymbol{\sigma}$ of a viscoplastic fluid belongs to an admissible convex set $K = \{\boldsymbol{\sigma} \in \mathbb{R}_S^{3\times3};\ |\boldsymbol{\sigma}'| \leq \kappa(p)\}$, where $p = -\,\mathrm{trace}(\boldsymbol{\sigma})/3$ is the pressure, $\boldsymbol{\sigma}' = \boldsymbol{\sigma} + p\boldsymbol{I}$ is the stress deviator and $\kappa = \kappa(p)$ is the yield limit. The boundary of K stands for the flow/no-flow condition. Conversely, if the stress is in K then the rate of deformation tensor $\boldsymbol{D}$ vanishes. If the stress tensor is not in K then we deal with an incompressible viscous flow described by the following constitutive equation:

$$\mathrm{trace}(\boldsymbol{D}) = 0, \quad \begin{cases} \boldsymbol{\sigma}' = 2\eta(\|\boldsymbol{D}\|, p)\boldsymbol{D} + \kappa(p)\dfrac{\boldsymbol{D}}{\|\boldsymbol{D}\|} & \text{if } \boldsymbol{D} \neq 0, \\ \|\boldsymbol{\sigma}'\| \leq \kappa(p) & \text{if } \boldsymbol{D} = 0, \end{cases} \tag{1}$$

where $\|\boldsymbol{A}\| =: |\boldsymbol{A}|/\sqrt{2} = \sqrt{\boldsymbol{A} : \boldsymbol{A}/2}$ denotes the second invariant of the deviator and η denotes the viscosity coefficient, which may depend on $|\boldsymbol{D}|$ and p, i.e. $\eta = \eta(|\boldsymbol{D}|, p)$. Note that the state of stress, $\boldsymbol{\sigma}'$, is represented as the sum of a viscous contribution $\boldsymbol{\sigma}^V = 2\eta(|\boldsymbol{D}|, p)\boldsymbol{D}$ (rate dependent) and a contribution $\boldsymbol{S} = \kappa(p)\frac{\boldsymbol{D}}{\|\boldsymbol{D}\|}$, related to plastic effects (rate independent). The viscous part of the stress $\boldsymbol{\sigma}^V$, as for a classical viscous fluid, is continuous in $\boldsymbol{D}$ and vanishes for $\boldsymbol{D} = 0$, i.e.

$$\|\boldsymbol{D}\|\eta(\|\boldsymbol{D}\|, p) \to 0, \quad \text{for} \quad \|\boldsymbol{D}\| \to 0. \tag{2}$$

At difference with the viscous contribution $\boldsymbol{\sigma}^V$, the plastic part $\boldsymbol{S}$ is not continuous in $\boldsymbol{D}$ and $\boldsymbol{S}$ does not vanish for $\boldsymbol{D} = 0$. For $\boldsymbol{D} \neq 0$ we get $|\boldsymbol{\sigma}'| = |\boldsymbol{D}|\eta(|\boldsymbol{D}|, p) + \kappa(p) > \kappa(p)$ and since (2) holds we obtain a continuous transition between flow and no-flow states (i.e. the flow rule and the non-flow condition are compatible).

For $\kappa(p) \equiv 0$ the plastic effects are vanishing and (1) reduces to a viscous fluid model. If η is independent of $|\boldsymbol{D}|$ and p, (1) reduces to the incompressible Navier–Stokes model but other choices can also be considered. For example, for $\eta(|\boldsymbol{D}|) = B\mathrm{arg\,sinh}\,(A|\boldsymbol{D}|)/|\boldsymbol{D}|$, with $A, B > 0$, we recover the Prandtl–Eyring type model and for $\eta = \mu|\boldsymbol{D}|^a$, with $a = 1/m - 1 > -1$, we deal with a power law (Norton model). Models of isotropic fluids have been used to describe the slow motions of soils and glaciers on natural slopes (see [40,1] for an overview on the subject).

The Von-Mises plasticity criterion (see Fig. 1 left):

$$\kappa(p) \equiv \kappa^0 > 0$$

was introduced to describe the plasticity of metals. If η is independent of $|\boldsymbol{D}|$ and p the constitutive Eq. (1) recover the classical *Bingham fluid* model (see [6]), used for many fluids with a solid like behavior (for instance oils or sediments in of oil drilling processes). This model, also denominated "Bingham solid" (see for instance [31]) was also considered to describe the (high rate) deformation of many solid materials having a fluid like behavior.

The plasticity (flow/no-flow) criterion

$$\kappa(p) = \kappa^0 + \mu p$$

(see Fig. 1 right) is called the Drucker–Prager model (see [13]). This yielding criterion was constructed as a simplification of Mohr–Coulomb plasticity: $\tau_{\max}(\boldsymbol{\sigma}') \leq C + p\tan(\phi)$ where $\tau_{\max}(\boldsymbol{\sigma}')$ is the Mohr–Coulomb tangential stress, C is the cohesion and ϕ is the angle of internal friction. The fact that the Mohr–Coulomb criterion is expressed in terms of the eigenvalues $\sigma_3 \leq \sigma_2 \leq \sigma_1$ of the stress tensor $\boldsymbol{\sigma}$ through $\sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)\sin(\phi) \leq 2C\cos(\phi)$, makes it very difficult to handle numerically. This is not the case for Drucker–Prager yield condition: $|\boldsymbol{\sigma}'| \leq \kappa_0 + \mu p$ which involves the norm of the deviator. The correspondence between the constitutive coefficients κ^0 and μ of the Drucker–Prager model and the coefficients of the Mohr–Coulomb model is not simple to establish. The usual choice is $\kappa^0 = C\cos(\phi)$, $\mu = \tan(\phi)$, but other choices can be found if one choose to reproduce different experimental settings. For the non-associate incompressible flow we have to put $\mu = \sin(\phi)$ in the in-plane case and $\mu = 6\sin(\phi)/(\sqrt{3}(3 - \sin(\phi)))$ for the triaxial compression (see for instance in [30]).

For constant viscosity η (not depending on $|\boldsymbol{D}|$ and p) we shall refer to the model as the "Drucker–Prager fluid". With an appropriate choice of the viscosity η the constitutive law (1) recovers the model proposed by Jop, Forterre and Pouliquen [24] for granular materials. To do that we have to put

$$\kappa^0 = 0, \quad \mu = \mu_s, \quad \eta(\|\boldsymbol{D}\|, p) = \frac{k(\mu_2 - \mu_s)p}{k\|\boldsymbol{D}\| + I_0\sqrt{p}},$$

where $k = d\sqrt{\rho_S}$ (with d is the diameter and ρ_S the density of the grains), I_0 an non-dimensional constant and μ_s, μ_2 are "friction coefficients".