Computers and Geotechnics 97 (2018) 111–123

Contents lists available at ScienceDirect

Computers and Geotechnics

journal homepage: www.elsevier.com/locate/compgeo

Research Paper

A high-order local artificial boundary condition for seepage and heat transfer

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ARTICLE INFO

Keywords:
Transient seepage
Heat transfer
Unbounded domain
Artificial boundarycondition
High-order approximation

ABSTRACT

A high-order local artificial boundary condition (ABC) with a new approximation scheme is proposed for numerical analyses of seepage and heat transfer in unbounded domains. The proposed ABCs are first derived for a one-dimensional case and then extended to high-dimensional cases and transversely isotropic media. In the derivation, the irrational function in Laplace space with respect to time is approximated through numerical integration. The calculations show that the proposed ABCs provide more satisfactory results than those obtained by using existing approximation methods, especially for long-duration simulations. Moreover, the relation among the calculation accuracy, approximation order, and diffusivity is also investigated.

1. Introduction

Physical problems in many fields, such as transient heat conduction and transient seepage, involve solving parabolic partial differential equations (PDEs). Geophysical problems, such as geothermal extraction, underground water flow and solute transport, all belong to this category of problems. The geometry of these problems can be regarded as a domain horizontally extending to infinity. To solve these practical problems by using numerical methods, the challenge of an infinite domain must be handled. A direct solution is to build a relatively large mesh, so that the outer boundary does not influence the entire area of analysis. However, these computations are inefficient and demand a high computational capacity. A typical approach is to truncate the infinite domain and to properly handle the truncated boundary; then, the problem can be simulated in a finite domain.

In the past several decades, considerable research was devoted to the treatment of the infinite domain. The methods proposed mainly include the infinite element method, artificial boundary method, and perfect matched layer (PML) method. As the method proposed in this paper belongs to the category of ABCs, the detailed introduction is limited to the ABCs. For the infinite element method and PML method, one can refer to [1,2] and [3,4], respectively.

The basic approach used to establish the ABCs is the determination of the explicit relationship between the space derivatives of the unknown variable with its time derivatives and the variable itself at the truncated boundary. In physics, this relationship provides the heat or water flux expressed by the unknown temperature or water head, respectively, for the heat conduction problem and seepage problem at the truncated boundary. The ABCs, according to its expression, can be divided into the global ABCs and the local ABCs. The former is a precise solution, which is usually expressed in integral form of the unknown variable or its derivatives of space and time. The current response on the artificial boundary is usually related to the responses of all previous time steps in the whole domain. Several representative methods such as the boundary element method [5,6] and the Dirichlet-to-Neumann (DtN) method [7,8] belong to this category. Methods of this type are inefficient to use because of the complicated calculation of the integrals. In contrast to the global ABC, the local ABC method is much easier to implement and more efficient to calculate since the local ABC method involves only the unknown variable and its derivatives at the time step and at the boundary point of interest. Though the local ABCs have these advantages, the main concern is their accuracy and numerical stability.

The ABCs have been applied to many kinds of problems. For dynamic problems, Lysmer and Kuhlemeyer [9] were recognized as the first to propose a viscous boundary. Deeks and Randolph [10] developed a viscous-spring boundary based on the analysis of a radially travelling wave in an axisymmetric plane. Li and Song [11–14] proposed several types of viscous-spring transmitting boundaries for wave propagation problems in unbounded, saturated poroelastic media and studied a three-dimensional numerical analysis for the longitudinal seismic response of tunnels under an asynchronous wave input by using the transmitting boundary. Regarding seepage or heat conduction problems, Han and Huang [15] derived a set of global ABCs for an axisymmetric heat conduction problem, and Carslaw and Jaeger [16] derived an integration form of ABCs for a one-dimensional problem. To

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https://doi.org/10.1016/j.compgeo.2018.01.008
Received 25 January 2017; Received in revised form 18 November 2017; Accepted 13 January 2018
Available online 03 February 2018
0266-352X/

apply these ABCs in a numerical calculation, Wu and Sun [17] established a finite differential scheme for one-dimensional global ABCs and proved that the scheme is unconditionally stable. Zheng [18] also considered the approximation and stability-related problems for a one-dimensional heat conduction problem. Chen and Song [19] used the idea of local ABCs for wave propagation to develop a simple but efficient local artificial boundary for transient seepage problems in an unbounded domain. Since this local ABC is based on a simple seepage problem with steady-state boundary conditions, the scope of the application is limited and the accuracy of the calculated results for problems with time-dependent boundary conditions are not quite satisfactory.

To obtain a more accurate calculation, high-order ABCs are used for dynamic problems [20] and heat conduction problems [21]. Wu and Zhang [21] proposed the high-order approximate ABCs for a heat transfer problem. The basic idea is to use a Laplace integral transformation and apply a Padé approximation to the variable s corresponding to time in Laplace space. By using the approximation, the ABCs in Laplace space can be easily inverted into time space, and the explicit expression of ABCs in time space is established. The accuracy of this high-order approximation approach mainly depends on the accuracy of the approximation for the variable s. For a long-duration analysis, a higher order approximation is needed. Otherwise, the accuracy of the solution is no longer satisfactory. To ensure a satisfactory solution, even for very long-duration simulations, a new approximation scheme is proposed to replace the Padé approximation as used by Wu and Zhang [21]. By using the new approximation scheme, a new high-order ABC with a higher accuracy is proposed.

The organization of this paper is as follows. Section 2 presents the high-order artificial boundary conditions for one-dimensional transient problems, and the derivation is extended to the multidimensional and for transversely isotropic media. A stability analysis for the one-dimensional case is also conducted. Section 3 focuses on the verification of the proposed high-order ABC for parabolic equations and demonstrates its superiority. Section 4 exhibits some numerical examples related to geothermal exploration, such as heat exchange in a single fracture, heat exchange in a well doublet system and seepage analysis in a doublet system, to validate the developed local ABCs. The numerical simulations are highly accurate and stable. Moreover, a comparative study is carried out to investigate the influence factors on the accuracy, especially the approximation order and the value of diffusivity. Computational efficiency-related issues are also discussed in Section 5 through the one-dimensional and two-dimensional cases, which reveal the advantages of using ABCs. Finally, discussions and conclusions are given in Section 6.

2. Derivation of the artificial boundary conditions

2.1. One-dimensional time-domain analysis

With reference to Wu and Zhang [21], it is straightforward to establish the time-domain global ABCs for one-dimensional seepage and heat transfer problems. For simplicity, a parabolic partial differential equation in a one-dimensional semi-infinite domain $x \in [0,+\infty]$ is first considered, and the unknown variable $u(x, t)$ is governed by

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} \tag{1}$$

$$u(x,0) = 0 \tag{2}$$

$$u|_{x=0} = u(0,t) \tag{3}$$

$$u \to 0, \quad \text{as } x \to \infty \tag{4}$$

For the heat conduction problem, where the unknown variable u is temperature, $a^2 = \kappa/(\rho c)$ is the thermal diffusivity, and κ is the thermal conductivity, ρc can be considered the volumetric heat capacity. For the transient seepage problem, u is the hydraulic head change, $a^2 = k/S_s$ is the hydraulic diffusivity, k is the hydraulic conductivity, and S_s is the water storage.

Applying the Laplace transformation with respect to t in Eq. (1), we have

$$s\tilde{u} = a^2\tilde{u}_{xx} \tag{5}$$

where the Laplace transformation is defined as

$$\tilde{u}(x,s) = \int_0^\infty e^{-st}u(x,t)dt \tag{6}$$

Eq. (5) has two linearly independent solutions in Laplace space, one is $\tilde{u}_1 = e^{x\sqrt{s/a^2}}$, and the other is $\tilde{u}_2 = e^{-x\sqrt{s/a^2}}$. Since the unknown variable u is bounded as $x \to +\infty$, the first solution should vanish from the actual solution for Eq. (5), and thus, the unknown variable u satisfies

$$\partial_x\tilde{u} + \sqrt{\frac{s}{a^2}}\tilde{u} = 0 \tag{7}$$

considering the following formula for the inverse Laplace transform

$$L^{-1}\left\{\frac{1}{\sqrt{s+\alpha}}\right\} = \frac{1}{\sqrt{\pi t}}e^{-\alpha t} \tag{8}$$

From Eqs. (7) and (8), the unknown variable $u(x,t)$ at any point x can be expressed as

$$\partial_x u + \sqrt{\frac{1}{a^2\pi}}\int_0^t \frac{1}{\sqrt{t-\tau}}\frac{\partial u}{\partial\tau}d\tau = 0 \tag{9}$$

Eq. (9) is the exact ABC at the artificial boundary for problems governed by Eqs. (1)–(4).

2.2. Approximation in Laplace space for the one-dimensional case

When using the ABCs in Eq. (9), in which the artificial boundary is global in time, special treatment is required. Wu and Sun [17] derived a stable discrete scheme in time space for Eq. (9), while Wu and Zhang [21] expanded the irrational function $\sqrt{z} = \sqrt{s/a^2}$ by using a Padé approximation

$$\sqrt{z} = \sqrt{\frac{s}{a^2}} = \sqrt{z_0}\sqrt{1+\left(\frac{z}{z_0}-1\right)} \approx \sqrt{z_0}-\sqrt{z_0}\sum_{k=1}^{N}\frac{b_k\left(1-\frac{z}{z_0}\right)}{1-a_k\left(1-\frac{z}{z_0}\right)} \tag{10}$$

where $a_k = \cos^2\left(\frac{k\pi}{2N+1}\right)$, $b_k = \frac{2}{2N+1}\sin^2\left(\frac{k\pi}{2N+1}\right)$, $k = 1, 2, \ldots, N$, and N is the approximate order. z_0 is the expansion point in the approximation in Eq. (10). In this paper, $z_0 = 1$, as proposed by Wu and Zhang [21].

The Padé approximation in Eq. (10) can guarantee an accurate approximation in a finite interval near the expansion point. However, the variable s in Laplace space with respect to t can vary widely. Then, the Padé approximation can no longer provide a satisfactory approximation over the entire time space; this result will be shown in the next section. To ensure accuracy over the entire time space, a new rational approximation in Laplace space, completely different from the Padé approximation, is proposed below.

It is clear that, for any variable s, we have [22]

$$\sqrt{s} = \frac{2s}{\pi}\int_0^\infty(\rho^2+s)^{-1}d\rho \tag{11}$$

Let $\rho = \tan\theta$; Eq. (11) can be converted to

$$\frac{2s}{\pi}\int_0^\infty(\rho^2+s)^{-1}d\rho = \frac{2s}{\pi}\int_0^{\pi/2}(\tan^2\theta+s)^{-1}\cos^{-2}\theta d\theta = \frac{2s}{\pi}\int_0^{\pi/2}\frac{\tan^2\theta+1}{\tan^2\theta+s}d\theta \tag{12}$$

From Eqs. (11) and (12) we have

$$\sqrt{s} = \frac{2s}{\pi}\int_0^{\pi/2}\frac{\tan^2\theta+1}{\tan^2\theta+s}d\theta \tag{13}$$