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Research Paper

Designing geotechnical structures with a proper stability criterion as a safety factor

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ABSTRACT

Many geotechnical structures are built on sandy soils. This complex medium exhibits a non-associated rate-independent behaviour. Owing to this pattern, local and global tangent operators become non-symmetric. Consequently, it has been proved that instabilities and failure may develop before reaching the classic failure limit given by a plasticity limit. A proper analysis with the second-order work criterion allows for a good description of these instabilities and can be used as a good failure criterion. In this paper, we review the main results obtained in the last few decades when this criterion was applied to homogeneous problems. Then, a numerical integration of this quantity and a method for its normalization are proposed. The results of this integration lead to the definition of a safety factor for a global structure even under non-homogeneous conditions. Finally, an application to the design of a nailed wall is proposed. In this framework, a constitutive model that gathers the main basic features of soil behaviour was developed. This model allows a given soil to be described with only one set of parameters, for example from a loose to a dense state or from a normally consolidated to an over-consolidated state. This feature is useful for taking into account initial states or for observing a change in the main behaviour due to a large change in the confining loading conditions.

1. Introduction

In this paper, we propose a method for analysing the stability of the equilibrium state of geomechanical systems. In most engineering applications, the behaviour of the soil can be described within the framework of elasto-plasticity theory. In the cases where the dimensions of the problems are sufficiently large, a classic finite element formulation can be used, but a complex constitutive model is required for the description of the behaviour of the soil. In this context, the failure of such systems is generally described with the help of a limit stress state or equivalently with a plasticity limit. Nevertheless, experimental evidence has shown that the failure of homogeneous samples of soils may occur strictly within the classic plasticity limit of Mohr–Coulomb [1–4]. In the last few decades, numerous studies have proved that Hill's stability criterion [5] describes these experimental facts properly [6–8]. Nevertheless, these works deal with homogeneous problems. In this article, we first review studies of the second-order work criterion from the works carried out in the last decade. This criterion is the local form under the small-strain assumption of Hill's criterion. Next, a numerical procedure is proposed for computing the second-order work over a whole volume, using a classic finite element formulation. A normalization of this quantity is proposed in order to use it as a safety factor to describe the failure of the system. Finally, an application to the design of a nailed wall is put forward. In this application, a constitutive model using the classic concepts of elasto-plasticity and gathering the main features of soil behaviour was developed. This model enables the description of a given soil with only one set of parameters whatever its density or consolidation state. This feature can be useful for describing the initial stress state of a problem or for describing complex physics, when large changes in confining stress occur during a particular loading path.

2. Background of the second-order work criterion

When considering an elasto-plastic medium subjected to dead loads on part of its surface and rigidly constrained on the remain-

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der, Hill [5] proved that a sufficient condition for stability of this medium is given by the following relationship in a Lagrangian formalism:

$$\int \left\{ \delta s_{ij} d\left(\frac{\partial u_j}{\partial X_i}\right)\right\} dV_0 > 0 \tag{1}$$

for any displacement du. In Eq. (1), the notations of the original paper have been kept; s_{ij} are the components of the nominal stress tensor, which is the transposed form of the non-symmetric Piolat-Lagrange (or identically Boussinesq) stress tensor; δs_{ij} is the change of s_{ij} due to the arbitrary virtual displacement $\underline{du}$. The integral on the elementary volume, is the sum of the inner quantity over the loading path between the initial configuration and the next infinitesimal configuration. The term on the left-hand side of the inequality is the amount of the internal work increment along the infinitesimal path. When it is positive, it means that the internal work of the medium is smaller than the external work applied. Consequently, this sufficient condition matches Lyapunov's definition of the stability of a system [9,6]. In the case of equilibrium states, this definition states that the response of the system remains bounded after a small disturbance. In fact when condition (1) is violated, internal and external works can be unbalanced and a deformation process can continue without any addition of external work.

In more recent works, Nicot [10] showed the link between the violation of condition (1) and an increase of the kinetic energy of the medium from a zero value. In the following expressions, σ_{ij}, $n_j dS$, and x_j are, respectively, the components of the Cauchy stress tensor, the unit external surface vector and the position vector in the actual configuration, while, $N_i dS_0$, and X_i are respectively the components of the unit external surface vector and the position vector in the initial configuration. Starting from a derivative of the kinetic energy theorem

$$\delta E_c(t) = \int_\Gamma \sigma_{ij} n_j \delta u_i dS - \int_V \sigma_{ij}\left(\frac{\partial(\delta u_i)}{\partial x_j}\right) dV \tag{2}$$

and a second-order Taylor development of the kinetic energy function with respect to time,

$$E_c(t+\delta t) = E_c(t) + \dot{E}_c(t)\Delta t + \ddot{E}_c(t)\frac{(\Delta t)^2}{2} + o(\Delta t)^3 \tag{3}$$

Nicot showed the following relationship:

$$2E_c(t+\delta t) = \int_{\Gamma_0} \delta s_{ij} N_i \delta u_j dS_0 - \int_{V_0} \delta s_{ij}\left(\frac{\partial(\delta u_j)}{\partial X_i}\right) dV_0 \tag{4}$$

This expression is only valid under the assumption that the sample is at equilibrium at time t, and that the kinetic energy function is at least of C_1 class. Thus, $E_c(t) = \dot{E}_c(t) = 0$, and derivatives of the internal and external powers are directly linked to $E_c(t+\delta t)$. Between t and $t+\delta t$, the system dynamically evolves only if $E_c(t+\delta t) > 0$. Under Hill's assumption, that the sample is subjected to dead loads on part of its surface and that kinematically fixed conditions are imposed on the rest of its surface, the term $\int_{\Gamma_0} \delta s_{ji} N_j \delta u_i dS_0$ is zero. Under this last condition we obtain:

$$2E_c(t+\delta t) = -\int_{V_0} \delta s_{ij}\left(\frac{\partial(\delta u_j)}{\partial X_i}\right) dV_0 \tag{5}$$

Consequently, when Hill's condition is violated, the kinetic energy of the system can evolve with a second-order rate with respect to time. We must not forget that Hill's condition is sufficient but not necessary to get a "burst" of kinetic energy. Physical conditions to get this "burst" are reviewed in the next sections.

We will now study the second-order work:

$$w_2 = \underline{\underline{d\sigma}} : \underline{\underline{d\varepsilon}} \tag{6}$$

This quantity represents the inner term in expression (1) when small strains and small geometrical changes are assumed, which is generally the case in engineering applications. $\underline{\underline{d\sigma}}$ is the Cauchy stress tensor and $\underline{\underline{d\varepsilon}}$ the linearised strain tensor. When considering homogeneous problems, Hill's condition becomes:

$$w_2 = \underline{\underline{d\sigma}} : \underline{\underline{d\varepsilon}} > 0 \tag{7}$$

Using notations where symmetric second-order tensors of strain and stress are written as a six-component vector, Eq. (7) can be written as follows:

$$w_2 = {}^t\underline{d\sigma}\ \underline{\underline{N}}\left(\underline{\tilde{d}}\right)\ \underline{d\sigma} > 0 \tag{8}$$

with $\underline{\underline{N}}\left(\underline{\tilde{d}}\right)$ the rate-independent constitutive operator which links $\underline{d\varepsilon}$ to $\underline{d\sigma}$ and ^{t} the transposed operator. This operator depends on the loading direction $\underline{\tilde{d}} = d\sigma/\|d\sigma\|$. In classical elasto-plastic models, this operator is piecewise linear in the stress rate space. We denote by *tensorial zone* a part of the stress rate space in which $\underline{\underline{N}}\left(\underline{\tilde{d}}\right)$ is linear, that is to say independent from $\underline{\tilde{d}}$. In such a tensorial zone, the following equation:

$$w_2 = {}^t\underline{d\sigma}\ \underline{\underline{N}}\ \underline{d\sigma} = 0 \Longleftrightarrow {}^t\underline{d\sigma}\ \underline{\underline{N_s}}\,\underline{d\sigma} = 0 \tag{9}$$

is the general equation of an elliptical cone. $\underline{\underline{N_s}}$ denotes the symmetric part of $\underline{\underline{N}}$. In the principal stress rate space, the solutions of Eq. (9) depend on eigenvalues of $\underline{\underline{N_s}}$ and are geometrically similar to the form displayed in Fig. 1 [11]. It is worth noting that the solutions of Eq. (9) appear in the order given in Fig. 1 along a given loading path. First, the solution of Eq. (9) is empty, because the soil sample is fully stable, then a unique unstable loading direction develops, and cones of unstable loading directions grow until the plasticity limit. The last mathematical solution given by the intersection of two planes has never been observed with our models. In fact it is the degenerate form of an elliptical cone with an infinite large axis. When a solution exists, loading paths included inside or on the cone are unstable, while other loading paths are stable. An illustration of such cones is displayed in Fig. 2. This result was obtained with Darve's octo-linear model [12], whose general form is given by:

$$\begin{bmatrix} d\varepsilon_1 \\ d\varepsilon_2 \\ d\varepsilon_3 \end{bmatrix} = \frac{1}{2}\left[N^+ + N^-\right]\begin{bmatrix} d\sigma_1 \\ d\sigma_2 \\ d\sigma_3 \end{bmatrix} + \frac{1}{2}\left[N^+ - N^-\right]\begin{bmatrix} |d\sigma_1| \\ |d\sigma_2| \\ |d\sigma_3| \end{bmatrix} \tag{10}$$

with

$$N^\pm = \begin{bmatrix} \frac{1}{E_1^\pm} & -\frac{\nu_{21}^\pm}{E_2^\pm} & -\frac{\nu_{31}^\pm}{E_3^\pm} \\ -\frac{\nu_{12}^\pm}{E_1^\pm} & \frac{1}{E_2^\pm} & -\frac{\nu_{32}^\pm}{E_3^\pm} \\ -\frac{\nu_{13}^\pm}{E_1^\pm} & -\frac{\nu_{23}^\pm}{E_2^\pm} & \frac{1}{E_3^\pm} \end{bmatrix} \tag{11}$$

N^+ is the tangent orthotropic matrix with respect to incremental loading, while N^- is the tangent orthotropic matrix with respect to incremental unloading. Such a model does not use classical assumptions of elasto-plasticity and in particular the assumption of splitting strain into an elastic and a plastic part. This model has been derived from a second-order Taylor expansion of the relation $\underline{\underline{N}}(\underline{\tilde{d}})\ \underline{d\sigma}$. This Taylor expansion first leads to the expression of Darve's incrementally non linear model (INL2) [12] and with further simplifications can lead to the octo-linear model.

The set of stress points where the solution of Eq. (9) is reduced to only one unstable direction is called the *bifurcation domain limit*. This limit is located inside the plasticity limit. Theoretically, this limit could depend on the loading path. Nevertheless, it is possible