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Research Paper

Determining an appropriate finite element size for modelling the strength of undrained random soils

J. Huang [a,*], D.V. Griffiths [a,b]

[a] ARC Centre of Excellence for Geotechnical Science and Engineering, The University of Newcastle, NSW, Australia
[b] Colorado School of Mines, Golden, CO, USA

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ABSTRACT

The mechanical properties of soils and rocks can be highly variable, and there has recently been a great deal of interest in modelling this variability using random field theory, in which the material properties vary from point to point. When these point-wise material properties are mapped onto a finite element mesh, discretization errors are inevitable. In this study, the discretization errors are studied and suggestions for element sizes in relation with spatial correlation lengths are given.

1. Introduction

Handling property variability is a research area of great importance and interest in civil, geotechnical and material engineering. At the micro scale, material properties are random due to spatially varied microstructures. Ideally, we would try to directly model this micro scale randomness to predict the overall macro scale performance, but this is obviously too computationally demanding to be practical. A compromise is to use a meso scale, i.e., the size of an element in a finite element method (FEM) simulation by homogenizing the micro scale randomness in each element to predict macro scale performance. When the meso scale is large enough to include all of the micro randomness, the overall material properties at the meso scale are spatially constant. For example, steel usually exhibits randomness on the micrometer scale, but typical finite elements with a size on the order of a centimeter are large enough to have constant properties. In geotechnical engineering, however, the material properties on the meso scale can show significant spatial variability. Random field theory is often used to model spatial variability. The starting point for a discussion of random field modelling is the "point" statistics that are assumed for the model. These are the hypothetical statistical properties of the soil or rock that would be measured if many tests could be performed on infinitesimal samples at a site or laboratory. A convenient measure of the spatial variability of a random field is the correlation length θ. Loosely speaking, θ is the distance within which points are significantly correlated (i.e., by more than approximately 10%). Conversely, two points separated by a distance greater than θ are largely uncorrelated. Many studies have been undertaken in recent years to develop probabilistic methods that address spatial variability in a systematic way (e.g., [9,19,11,13,14,16,1,21,18,17,20]). Of particular importance has been the development of the random finite element method (RFEM) for modelling the spatial variability of geomaterials (e.g., [7]). While mesh effects have been investigated for highly variable materials by some investigators (e.g., [8,5,6,13,14,2,4,15]), the discretization error due to the element size has received little formal attention; the work of with Ching and Phoon [3] is an exception. A small correlation length means that the properties of a soil change rapidly from place to place; therefore, the element size should be small enough to capture the spatial randomness of the material properties. The major aim of the present work, therefore, is to establish a more formal link between the maximum element size and the spatial correlation length. The paper begins with a review of the local averaging method based on the geometric mean. The local averaging method can provide an analytical estimate of the effective overall property by taking the spatial correlation structure into account. Although the geometric mean is dominated by low values, it ignores the "seeking out" effect of a failure mechanism. This effect will be examined using direct Monte Carlo simulations. From the direct simulations, the discretization errors are studied and suggestions for appropriate element sizes in relation with spatial correlation lengths are given.

* Corresponding author.

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2. Modelling the strength of materials using local averaging

For the purpose of demonstration, this study will focus on the strengths of engineering materials. It is assumed that the strengths of materials are modelled by point-wise random fields. The question is how small the element size should be for a given spatial correlation length. We restrict ourselves to isotropic Gaussian random fields or random fields that can easily be transformed into Gaussian random fields. We want to investigate the strength in a certain domain (i.e., the macroscale strength) in which the strength at any micro-scale point within the domain is modelled by a random field. In this section, the overall strength of a certain domain is estimated analytically using local averaging. The results will be compared to those of RFEM simulations.

The overall strength of a material in a certain domain is usually dominated by its low-strength regions. Because it is dominated by low values, the geometric mean is recommended by Fenton and Griffiths [7] for estimating the overall strength. The geometric mean is defined as the nth root of the product of n (nonnegative) random variables. Using this definition, the discrete set of random variables $X_1, X_2, \ldots, X_n$ has the geometric mean

$$X_G = (X_1, X_2, \ldots, X_n)^{1/n}. \tag{1}$$

X_G weights low values more heavily than high values (low values dominate). This can be seen by considering what happens to the geometric mean (see Eq. (1)) if even a single X_i is zero: X_G becomes zero.

By expressing Eq. (1) as a power of e, we obtain an alternative way of computing the geometric mean,

$$X_G = \exp\left(\frac{1}{n}\sum_{i=1}^{n} \ln X_i\right). \tag{2}$$

If X is a 1D continuously varying spatial random field, the geometric mean of X over a given domain becomes

$$X_G = \exp\left(\frac{1}{R}\int_0^R \ln X(\xi)d\xi\right), \tag{3}$$

where R is the length over which X is averaged and ξ is a spatial coordinate.

If X is log-normally distributed, X_G tends to be log-normally distributed, according to the central limit theorem. The mean and variance of X_G are found by first finding the mean and variance of $\ln X_G$. The mean of $\ln X_G$ is

$$\begin{aligned} E[\ln X_G] &= E[\tfrac{1}{R}\textstyle\int_0^R \ln X(\xi)d\xi] \\ &= \tfrac{1}{R}\textstyle\int_0^R E[\ln X(\xi)]d\xi \\ &= E[\ln X] \\ &= \mu_{\ln X} \end{aligned} \tag{4}$$

where $\mu_{\ln X}$ is the mean of $\ln X$,

$$\mu_{\ln X} = \ln \mu_X - \frac{1}{2}\sigma^2_{\ln X} \tag{5}$$

and

$$\sigma_{\ln X} = \sqrt{\ln\left(1 + \left(\frac{\sigma_X}{\mu_X}\right)^2\right)}, \tag{6}$$

where μ_X and σ_X are the mean and standard deviation of a log-normally distributed random field X .

We note that because the median of a log-normally distributed random field X is $\exp(\mu_{\ln X})$, the median of X_G is equal to the median of X. In other words, taking the geometric average of a log-normally distributed random field X preserves both the type of distribution and its median.

The variance of $\ln X_G$ is

$$\begin{aligned} \mathrm{Var}\,[\ln X_G] &= E\left[\tfrac{1}{R}\textstyle\int_0^R [\ln X(\xi) - \mu_{\ln X}]d\xi \tfrac{1}{R}\int_0^R [\ln X(\eta) - \mu_{\ln X}]d\eta\right] \\ &= \tfrac{1}{R^2}\textstyle\int_0^R\int_0^R E[(\ln X(\xi) - \mu_{\ln X})(\ln X(\eta) - \mu_{\ln X})]d\xi d\eta \\ &= \tfrac{1}{R^2}\textstyle\int_0^R\int_0^R C_{\ln X}(\tau)(\xi - \eta)d\xi d\eta \\ &= \tfrac{\sigma^2_{\ln X}}{R^2}\textstyle\int_0^R\int_0^R \rho_{\ln X}(\tau)(\xi - \eta)d\xi d\eta \\ &= \sigma^2_{\ln X}\gamma_{\ln X}(R) \end{aligned} \tag{7}$$

where ζ and η are spatial coordinates, τ is the distance between two points, $C_{\ln X}(\tau)$ is the covariance function of $\ln X$, $\rho_{\ln X}(\tau)$ is the correlation function of $\ln X$ such that $C_{\ln X}(\tau) = \sigma^2_{\ln X}\rho_{\ln X}(\tau)$, $\sigma_{\ln X}$ is the mean and standard deviation of $\ln X$, and $\gamma_{\ln X}(R)$ is the variance function that determines how much the variance is reduced when X is averaged over a length R using Eq. (1),

$$\gamma_{\ln X}(R) = \frac{1}{R^2}\int_0^R\int_0^R \rho_{\ln X}(\tau)(\xi - \eta)d\xi d\eta \tag{8}$$

There are a few commonly used correlation functions (see, e.g., [7]). The Markov correlation function used in this study is

$$\rho_{\ln X}(\tau) = \exp\left(-\frac{|\tau|}{\theta_{\ln X}}\right), \tag{9}$$

where $\theta_{\ln X}$ is the spatial correlation length of $\ln X$.

The correlation function in logarithmic space can be converted to the following correlation function in real space (e.g., [25]):

$$\rho_X(\tau) = \frac{\exp(\sigma^2_{\ln X}\rho_{\ln X}(\tau)) - 1}{\exp(\sigma^2_{\ln X}) - 1} \tag{10}$$

For most random fields, the two correlation functions are quite similar and

$$\theta_{\ln X} = \theta_X \tag{11}$$

Because of Eq. (11), from now on, the spatial correlation lengths in the logarithmic and real spaces are both denoted by θ.

In 2D, for a rectangular domain with side lengths R_x and R_y, the variance function is defined by

$$\gamma_{\ln X}(R_x, R_y) = \frac{4}{R_x^2 R_y^2}\int_0^{R_y}\int_0^{R_x} (R_x - x)(R_y - y)\rho_{\ln X}(\tau)\,dx\,dy. \tag{12}$$

Using the correlation function given in Eq. (9), the variance function can be obtained analytically as follows:

$$\gamma_{\ln X}(R_x, R_y) = \frac{4\theta^4}{R_x^2 R_y^2}\left[\frac{R_x}{\theta} + \exp\left(-\frac{R_x}{\theta}\right) - 1\right]\left[\frac{R_y}{\theta} + \exp\left(-\frac{R_y}{\theta}\right) - 1\right], \tag{13}$$

where R_x and R_y are the lengths of the sides of a rectangular domain.

Once the mean and variance of $\ln X_G$ have been computed, the mean and variance of X_G can be computed using

$$\begin{aligned} \mu_{X_G} &= \exp\left(\mu_{\ln X} + \tfrac{1}{2}\sigma^2_{\ln X}\gamma_{\ln X}\right) \\ &= \frac{\mu_X}{\sqrt{\left(1 + \left(\frac{\sigma_X}{\mu_X}\right)^2\right)^{1-\gamma_{\ln X}}}} \end{aligned} \tag{14}$$

and

$$\begin{aligned} \sigma_{X_G} &= \mu_{X_G}\sqrt{\exp(\sigma^2_{\ln X}\gamma_{\ln X} - 1)} \\ &= \mu_{X_G}\sqrt{\exp\left(\ln\left(1 + \left(\frac{\sigma_X}{\mu_X}\right)^2\right)\gamma_{\ln X} - 1\right)} \end{aligned} \tag{15}$$