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Research Paper

# Analysis of axisymmetric thermo-elastic problem in multilayered material with anisotropic thermal diffusivity

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ABSTRACT

An analytical solution is derived for the axisymmetric thermo-elastic problem of multilayered material with anisotropic thermal diffusivity due to a buried heat source. By applying the Laplace–Hankel transform to the state variables involved in the basic governing equations, the analytical layer-element which describes the relationship between the transformed generalized stresses and displacements is obtained. Considering the continuity conditions between adjacent layers and the boundary conditions, the global stiffness matrix for a multilayered system is assembled and solved in the transformed domain. The actual solutions of the problem in the physical domain are acquired by inverting the Laplace–Hankel transform. Finally, some numerical examples are given to demonstrate the accuracy of the proposed method and to illustrate the influences of the heat source's types and the anisotropy of thermal diffusivity on the thermo-elastic response.



## 1. Introduction

The thermo-elastic problem of the time-dependent behavior of material containing a heat source is of significant in environmental engineering and civil engineering. The heat source such as a canister of radioactive waste is usually deposited at a large depth, such as 100–700 m below ground, to avoid affecting humans. The heat source continuing to generate heat for a long period of time may lead to temperature rising and volume expanding of the surrounding media. Therefore, extensive attention has been received on the problem in the field of geology, environmental engineering, soil science and civil engineering.

The constitutive equations for thermo-elastic material, which express the relations between the stress, the strain and the temperature change, were first introduced by Biot [1]. With Biot's theory, many solutions for thermal response caused by the change of temperature have been developed by numerous investigators. Keramidas and Ting [2] developed two finite element models to deal with the problem of induced thermal stresses in a medium subjected to temperature changes on its boundary surface. Ghosna and Sabbaghianb [3] investigated axisymmetric quasi-static coupled problems of thermoelasticity for cylindrical regions with the aid of Laplace transform. Small and Booker [4,5] presented the solutions for the behavior of layered soil or rock deposits which contain a heat source with the help of the finite layer method. Carter and Booker [6] proposed a finite element method to solve the governing equations for the fully coupled theory of thermoelasticity, which revealed that semi-coupled theory could provide sufficient accuracy in most geotechnical problems. Sharma and Chand [7] investigated the axisymmetric and plane strain problems in generalized theories of thermo-elasticity by employing the eigenvalue approach after application of the Laplace and Hankel transforms. The boundary element method was developed for problems of quasi-static axisymmetric thermo-elasticity by Dargush and Banerjee [8]. Brock et al. [9] obtained fundamental thermoelastic two-dimensional solutions for thermal and/or mechanical loadings moving unsteadily over the surface of a half-space. Considering the theory of thermoelasticity, Sherief and Megahed [10] solved the two-dimensional problem for a half space whose surface is traction free and subjected to the effects of heat sources. Zhong and his coworkers [11,12] applied the transfer matrix method to solve the thermal stress problem with variable temperature of a multilayered elastic half-space system. As we know from practice, the typical deposit process of natural geomaterials may lead to clear differences in thermal diffusivity between different directions, especially for the horizontal and vertical thermal diffusivity. Therefore, some researchers focused

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their attention on the study of the thermo-elastic response of the material with anisotropic thermal diffusivity [13,14]. Yue [13] provided an analytical solution for the fundamental transient thermo-elastic problem with body forces and a heat source in vertically inhomogeneous media. By utilizing a similar approach to Sharma and Chand [7], Sharma and Kumar [14] investigated the plane strain problems in generalized theory of thermo-elasticity in a homogeneous transversely isotropic medium.

The objective of this paper is to introduce the analytical layer-element method [15–18] to study the behavior of multilayered material with anisotropic thermal diffusivity containing a heat source at an arbitrary depth. Three types of heat sources, a point heat source, a circular area source and a ring heat source, are considered to satisfy different practices. Based on the basic governing equations of the discussed problem in elasticity and heat transfer, the analytical layer element, which establishes the relationship between the generalized displacements and stresses for a single material layer, is acquired in the transformed domain with the help of the Laplace–Hankel transform. Then, the global stiffness matrix equation of the multilayered material is further obtained by assembling the interrelated layer elements, according to the continuity conditions between adjacent layers. The solutions in the Laplace–Hankel transformed domain are obtained by solving the global stiffness matrix equation satisfying the boundary conditions, and the actual solutions in the physical domain can be acquired by the inversion of the Laplace–Hankel transform. Finally, based on the analytical solutions, numerical examples are presented to illustrate the correctness of the method, and to investigate the effects of different material parameters on temperature increment and variation of displacement.

## 2. Governing equations

In the absence of body forces, the stresses in the cylindrical axisymmetric coordinate system should satisfy the equilibrium equations:

$$\frac{\partial \sigma_r}{\partial r}+\frac{\partial \sigma_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}=0 \tag{1a}$$

$$\frac{\partial \sigma_{rz}}{\partial r}+\frac{\partial \sigma_{z}}{\partial z}+\frac{\sigma_{rz}}{r}=0 \tag{1b}$$

where $\sigma_r$, $\sigma_\theta$ and $\sigma_z$, are the normal stress components in $r$, $\theta$ and $z$ directions, respectively; $\sigma_{rz}$ is the shear stress component in the plane $r$–$z$.

The stress–strain relationship of thermo-elastic medium can be written as [1]:

$$\boldsymbol{\sigma}+\frac{\alpha E}{1-2\mu}T\boldsymbol{\vartheta}=\lambda e\boldsymbol{\vartheta}+2G\boldsymbol{\varepsilon} \tag{2}$$

where $\boldsymbol{\sigma}=(\sigma_r,\sigma_\theta,\sigma_z,\sigma_{rz})^{\mathrm{T}}$ is the vector of stress components; $\boldsymbol{\varepsilon}=\left(\frac{\partial u_r}{\partial r},\frac{u_r}{r},\frac{\partial u_z}{\partial z},\frac{\partial u_r}{2\partial z}+\frac{\partial u_z}{2\partial r}\right)^{\mathrm{T}}$ is the vector of strain components, here $u_r$ and $u_z$ represent the displacement components in $r$ and $z$ directions, respectively; $\boldsymbol{\vartheta}=(1,1,1,0)^{\mathrm{T}}$; $e=\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z}$ denotes the dilatation; $\lambda=\frac{2G\mu}{1-2\mu}$ and $G=\frac{E}{2(1+\mu)}$ are the Lamé modulus and elastic shear modulus of the material, in which $E$ and $\mu$ denote Young's modulus and Poisson's ratio of the material, respectively; $\alpha$ represents the coefficient of linear thermal expansion; and $T$ denotes the temperature increment.

Combining Eqs. (1) and (2), we have:

$$\nabla^2 u_r+\frac{1}{1-2\mu}\frac{\partial e}{\partial r}-\frac{1}{r^2}u_r-\frac{2\alpha(1+\mu)}{1-2\mu}\frac{\partial T}{\partial r}=0 \tag{3a}$$

$$\nabla^2 u_z+\frac{1}{1-2\mu}\frac{\partial e}{\partial z}-\frac{2\alpha(1+\mu)}{1-2\mu}\frac{\partial T}{\partial z}=0 \tag{3b}$$

where $\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}$ is the Laplacian operator for the axisymmetric thermo-elastic problem.

The flow of heat within the media is assumed to be governed by Fourier's law of heat conduction:

$$\boldsymbol{q}=-\boldsymbol{\lambda}_{\mathbf{T}}\nabla T \tag{4a}$$

where $\boldsymbol{q}=(q_r,q_z)^{\mathrm{T}}$ is the vector of heat flow, here $q_r$, $q_z$ are the heat flow in the $r$ and $z$ directions, respectively; $\nabla=(\partial/\partial r,\partial/\partial z)^{\mathrm{T}}$ is the gradient operator; $\boldsymbol{\lambda}_{\mathbf{T}}=(\lambda_{Tr},\lambda_{Tz})^{\mathrm{T}}$ denotes the vector of the coefficient of heat conductivity, and $\lambda_{Tr}$, $\lambda_{Tz}$ are the coefficients of heat conductivity in the $r$ and $z$ directions, respectively.

The total heat flow $Q$ in the $z$ direction within the time from 0 to $t$ can be written as follows:

$$Q=\int_0^t q_z \mathrm{d}t \tag{4b}$$

By consideration of the semi-coupling of the elastic and thermal processes, the heat diffusion equation with anisotropic thermal diffusivity can be expressed as:

$$\frac{\partial T}{\partial t}=a_r\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}\right)+a_z\frac{\partial^2 T}{\partial z^2} \tag{5}$$

where $a_r=\lambda_{Tr}/c\rho$ and $a_z=\lambda_{Tz}/c\rho$ are the horizontal and vertical coefficients of thermal diffusivity, respectively, here $\rho$ and $c$ represent the density and the specific heat of the material, respectively.

## 3. Derivation of the analytical layer-element

The $m$th-order Laplace–Hankel transform with respect to the variables $t$ and $r$ are defined as [19]:

$$\bar{f}(\xi,z,s)=\int_0^\infty\int_0^\infty f(r,z,t)J_m(\xi r)r\mathrm{e}^{-st}\mathrm{d}r\mathrm{d}t \tag{6a}$$

$$f(r,z,t)=\frac{1}{2\pi \mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}\int_0^\infty \bar{f}(\xi,z,s)J_m(\xi r)\xi\mathrm{e}^{st}\mathrm{d}\xi\mathrm{d}s \tag{6b}$$

where $s$ and $\xi$ are the Laplace and Hankel transform parameters, respectively; $\mathrm{i}=\sqrt{-1}$; $J_m(\xi r)$ denotes the $m$th-order Bessel function of the first kind.

Applying the operators $\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)$ and $\frac{\partial}{\partial z}$ to Eqs. (3a) and (3b), respectively, we obtain:

$$\nabla^2\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)u_r+\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\left(\frac{e}{1-2\mu}-\frac{2\alpha(1+\mu)}{1-2\mu}T\right)=0 \tag{7a}$$

$$\nabla^2\left(\frac{\partial u_z}{\partial z}\right)+\frac{\partial^2}{\partial z^2}\left(\frac{e}{1-2\mu}-\frac{2\alpha(1+\mu)}{1-2\mu}T\right)=0 \tag{7b}$$

Combining Eqs. (7a) and (7b):

$$\nabla^2(e-\alpha\eta T)=0 \tag{8}$$

where $\eta=\frac{1+\mu}{1-\mu}$.

Applying the zeroth-order Laplace–Hankel transform to Eqs. (5) and (8), we have:

$$s\overline{T}=-a_r\xi^2\overline{T}+a_z\frac{\mathrm{d}^2\overline{T}}{\mathrm{d}z^2} \tag{9a}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-\xi^2\right)(\bar{e}-\alpha\eta\overline{T})=0 \tag{9b}$$

Solving Eqs. (9a) and (9b) yields:

$$\overline{T}=\mathrm{e}^{-zk}A_1+\mathrm{e}^{zk}A_2 \tag{10a}$$

$$\bar{e}=\alpha\eta\mathrm{e}^{-zk}A_1+\alpha\eta\mathrm{e}^{zk}A_2+\mathrm{e}^{-z\xi}A_3+\mathrm{e}^{z\xi}A_4 \tag{10b}$$