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Upper bound finite element analysis of slope stability using a nonlinear failure criterion

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ABSTRACT

In this study, upper bound finite element (FE) limit analysis is applied to stability problems of slopes using a nonlinear criterion. After formulating the upper bound analysis as the dual form of a second-order cone programming (SOCP) problem, the stress field and corresponding shear strength parameters can be determined iteratively. Thus, the nonlinear failure criterion is represented by the shear strength parameters associated with stress so that the analysis of slope stability using a nonlinear failure criterion can be transformed into the traditional upper bound method with a linear Mohr–Coulomb failure criterion. Comparison with published solutions illustrates the accuracy and feasibility of the proposed method for a simple homogeneous slope stability problem. The proposed approach is also applied to a seismic stability problem for a rockfill dam to study the influence of different failure criterions on the upper bound solutions. The results show that the seismic stability coefficients obtained using two different nonlinear failure criteria are similar but that the convergence differs significantly.

1. Introduction

The analysis of slope stability is a very important issue in geotechnical engineering projects. While the limit equilibrium method has traditionally been used to assess the stability of slopes [1,2], many researchers are attempting to elaborate and develop new calculation methods. The limit analysis method [3], which involves collapse load calculations using the lower and upper bound theorems of plasticity, has provided engineers with a convenient alternative for slope stability problems. In recent years, numerical lower and upper bound techniques in conjunction with the finite element method (FEM) have been powerful tools for stability problems because of their rigorous lower and upper bound solutions. Sloan [4,5], Sloan and Kleeman [6], Yu et al. [7], and Lyamin and Sloan [8,9] have made significant progress in developing the finite element (FE) limit analysis using linear programming (LP) or nonlinear programming (NLP). Recent research by Makrodimopoulos and Martin [10–12] has concentrated on FE limit analysis using second-order cone programming (SOCP) to solve stability or bearing capacity problems.

Regardless of whether LP, NLP, or SOCP is used in limit analysis, geomaterials are commonly assumed to obey a linear Mohr–Coulomb failure criterion for slope stability problems. However, a substantial number of experiments have clearly shown that the failure criterion of almost all geomaterials has a nonlinear nature [13–16]. At present, there are two types of methods that are suitable for upper bound analysis using a nonlinear failure criterion. One of them is the inverse method developed by Zhang and Chen [17], which is based on a variational calculus technique. The other is the tangent method proposed by Drescher and Christopoulos [18] and further surveyed by Yang and Yin [19]. According to the tangent method, the nonlinear failure envelope is represented by a linear Mohr–Coulomb failure criterion with a straight tangential line, which provides an upper bound solution due to the convexity of the failure envelope. However, in this method, the friction angle is assumed to be constant along the length of the sliding surface, and the normal and shear stresses must also be constant for a nonlinear material [20]. These constant values do not conform to the actual stress distribution of slopes.

According to plasticity limit theorems, it is known that a kinematically admissible velocity field will be constructed by the upper bound method and a statically admissible stress field will be constructed by the lower bound method. When a nonlinear failure criterion is used in the upper bound limit analysis of slope stability, the main problem is how to represent the shear strength parameters, which are commonly expressed in terms of stress. In fact, the duality between the upper and lower bound theorems of plasticity has an analogy in the duality theory of mathematical programming [21]. On the other hand, the upper and lower bound theorems in the limit theorem are dual to each other. This means we can get the stress field that satisfies the conditions of equilibrium, the flow rule and boundary in a weak sense from the dual of upper bound analysis. For this reason, if the dual problem of the upper bound

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analysis can be formulated and optimised, the stress field will be obtained, which can be used to represent the nonlinear failure criterion exactly in slope stability analysis.

Based on the recent work by Makrodimopoulos and Martin [11], this paper develops an upper bound FEM to analyse slope stability using a nonlinear failure criterion. First, the dual problem of the upper bound limit analysis is formulated and optimised by means of a six-node triangle element without discontinuities. Then, the stress distribution of slopes and the corresponding equivalent shear strength parameters can be determined iteratively to obtain the rigorous upper bound solutions of the slope stability factor and the corresponding displacement fields. As distinct from the tangent method, the nonlinear failure envelope is no longer represented by a straight tangential line but by the equivalent shear strength parameters associated with the stress distribution of slopes.

2. Upper bound theorem

Limit analysis consists of two theorems, namely, the upper and lower bound theorems. The upper bound theorem states that the external loads obtained are not lower than the true collapse loads if a kinematically admissible displacement field can be found with the conditions of boundary, the flow rule, and the energy–work balance equation. Makrodimopoulos and Martin [11] demonstrated that the six-node triangular element without a discontinuous displacement field can be used to obtain strict upper bounds. Consider a rigid, perfectly plastic construction V with boundary S; according to the upper bound theorem, when the structure collapses, there exists a kinematically admissible displacement field such that energy dissipation is no more than the work of external force:

$$\int_V \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u})\mathrm{d}V \leqslant \mathbf{q}^{\mathrm{T}}\mathbf{u} \quad \forall \boldsymbol{\sigma} \in F = \{\boldsymbol{\sigma} | f(\boldsymbol{\sigma}) \leqslant 0\} \tag{1}$$

where $\mathbf{q}$ are equivalent nodal loads, $\mathbf{u}$ should satisfy the boundary conditions of S, and $\mathbf{u} = \mathbf{u}_0$. Because displacement discontinuities between elements do not exist, the power to be dissipated by plastic deformation is only permitted to occur within each triangular element. The plastic dissipation function is defined as

$$d_{\mathrm{p}}(\boldsymbol{\varepsilon}) = \sup_{\boldsymbol{\sigma} \in F} \sum \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \tag{2}$$

and the set of plastically admissible strains is

$$E = \left\{\boldsymbol{\varepsilon} : d_{\mathrm{p}}(\boldsymbol{\varepsilon}) < +\infty\right\} \tag{3}$$

The plastically admissible strains are those that satisfy the associated flow rule. We divide equivalent nodal loads into two parts: collapse loads $\mathbf{q}_1$, which are subjected to a multiplier factor β, and constant loads $\mathbf{q}_0$, which are not subjected to β. Using the definition in Eq. (2), the energy–work balance condition in Eq. (1) can be written as

$$D_{\mathrm{p}}(\boldsymbol{\varepsilon}) \leqslant \boldsymbol{\beta}\mathbf{q}_1^{\mathrm{T}}\mathbf{u} + \mathbf{q}_0^{\mathrm{T}}\mathbf{u} \quad \text{and} \quad \boldsymbol{\varepsilon}(\mathbf{u}) \in E \tag{4}$$

where

$$D_{\mathrm{p}}(\boldsymbol{\varepsilon}) = \int_V d_{\mathrm{p}}(\boldsymbol{\varepsilon})\mathrm{d}V \tag{5}$$

Thus, an upper bound on β can be calculated by solving the following optimisation problem:

$$\begin{aligned} \min \quad & D_{\mathrm{p}}(\boldsymbol{\varepsilon}) - \mathbf{q}_0^{\mathrm{T}}\mathbf{u} \\ \text{s.t.} \quad & \boldsymbol{\varepsilon}(\mathbf{u}) \in E \text{ in } V \\ & \mathbf{u} = \mathbf{u}_0 \text{ on } S \\ & \mathbf{q}_1^{\mathrm{T}}\mathbf{u} = 1 \end{aligned} \tag{6}$$

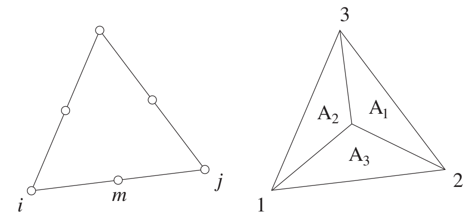

Fig. 1. Six-node linear strain element for upper bound analysis.

3. Elements used for upper bound analysis

The six-node triangle finite element with straight sides that is used in the upper bound analysis is shown in Fig. 1. Noted that a side i–j–m of the six-node element in Fig. 1 will only be straight if

$$x_m = (x_i + x_j)/2 \tag{7}$$

Because the nodal displacement field can be expressed as a quadratic form within a triangular element, any strain component may vary linearly. Moreover, if the element sides are straight, the strain tensor at any point within the triangle can be expressed as a combination of those at three vertices, i.e.,

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \sum_{i=1}^{3} L_i(\mathbf{x})\varepsilon_i, 0 \leqslant L_i(\mathbf{x}) \leqslant 1, \sum_{i=1}^{3} L_i(\mathbf{x}) = 1 \tag{8}$$

where the coefficients $L_i = A_i/(A_1 + A_2 + A_3)$ are area coordinates. Because any strain tensor can be expressed by ε_i at the vertices, if the flow rule is enforced at the three vertices of the element, it holds at all points within the elements. Thus, the flow rule constraint only needs to be enforced at a finite number for it to hold throughout the structure.

4. Mohr–Coulomb criterion in plane strain conditions

As mentioned in Ref. [11], the main restriction of the proposed method is that the yield function must be expressed as a conic quadratic form. The Mohr–Coulomb criterion in plane strain conditions is a typical example of a yield restriction with a conic quadratic form. It can be written in the following form:

$$\|\mathbf{s}^{\mathrm{red}}\| + a\sigma_m - k \leqslant 0 \tag{9}$$

where

$$\|\mathbf{s}^{\mathrm{red}}\| = [s_{11} \quad s_{12}]^{\mathrm{T}} \tag{10}$$

$$\sigma_m = \frac{1}{D}\sum_{i=1}^{D} \sigma_{ii}, s_{ij} = \sigma_{ij} - \sigma_m \delta_{ij} \tag{11}$$

where D is the dimension of the tensors and δ is Kronecker's δ. For plane strain conditions of the Mohr–Coulomb criterion, $D = 2$, $a = \sin\varphi$, and $k = c\cos\varphi$, where c is the cohesion and φ is the internal friction angle. Using the definition in Eq. (2) and the upper bound theorem, the dissipation function for plastically admissible strains can be derived to the following form for any $a \geqslant 0$ [11]:

$$d_{\mathrm{p}} = k\lambda \text{ with } \theta = a\lambda \text{ and } \lambda \geqslant \|\mathbf{e}^{\mathrm{red}}\| \tag{12}$$

where λ is an auxiliary variable and θ is the volume expansion as

$$\theta = \sum_{i=1}^{D} \varepsilon_{ii} = \mathrm{div}\mathbf{u} \tag{13}$$

$$\mathbf{e}^{\mathrm{red}} = [2e_{11} \quad 2e_{12}]^{\mathrm{T}}, \quad e_{ij} = \varepsilon_{ij} - \frac{1}{D}\theta\delta_{ij} \tag{14}$$