Computers and Geotechnics 48 (2013) 127–133

Contents lists available at SciVerse ScienceDirect

Computers and Geotechnics

journal homepage: www.elsevier.com/locate/compgeo

Exact solutions for the one-dimensional transient response of unsaturated single-layer porous media

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ARTICLE INFO

Article history:
Received 19 January 2012
Received in revised form 9 August 2012
Accepted 11 August 2012
Available online 12 December 2012

Keywords:
Poromechanics
Exact solution
Unsaturated
Transient response

ABSTRACT

The one-dimensional transient response of unsaturated single-layer porous media is studied based on the theory of unsaturated porous media proposed by Zienkiewicz et al., and exact time-domain solutions are obtained for three types of nonhomogeneous boundary conditions. During the solution procedure, the nonhomogeneous boundary conditions are transformed into homogeneous boundary conditions. Then, the eigenfunction expansion method is utilised to obtain the exact solutions for these new boundary conditions. Several numerical examples are provided to investigate the propagation of compressional waves, and it is verified that three types of compressional waves exist in unsaturated porous media that contain two immiscible fluids.

1. Introduction

Wave motion in porous media is an important issue in many fields, such as geotechnical engineering and biological engineering. A porous medium is considered saturated if it is filled with a single type of fluid and considered unsaturated if it is filled with multiple fluids. In most cases, two fluids coexist within the porous space: either two liquids, such as oil and water in petroleum engineering applications, or a liquid and a gas, such as water and air in geotechnical engineering applications [1].

Since the Biot theory [2] was established, scholars have made great efforts to study the mechanisms of wave propagation and attenuation in porous media through numerical and analytical methods [3,4]. However, due to the complexity of the inertial, viscous, and mechanical couplings in porous media, most dynamic problems can only be solved quantitatively via numerical methods involving the discretisation of both spatial and temporal domains [4]. Even for the one-dimensional problem, only a few analytical solutions are available in the literature.

For saturated porous media, due to the difficulty in mathematics, only several analytical solutions for the one-dimensional transient response are available in the literature. Using the Laplace transform method, Garg et al. [5], Simon et al. [6], and de Boer et al. [7] obtained analytical solutions for a semi-infinite porous medium in several special cases. Gajo and Mongiovi [8] obtained the general solution for a single-layer porous medium and the analytical solution for the step displacement boundary condition through the eigenfunction expansion method. By using the Laplace transform method, Schanz and Cheng [9] gave the analytical solution for a single-layer problem by ignoring the viscous coupling effect. Shan et al. [10] used the eigenfunction expansion method and presented three exact solutions for a single-layer porous medium considering three types of boundary conditions.

However, compared to the research on saturated porous media, few studies have been concerned with the one-dimensional transient response of unsaturated porous media. As far as we know, an exact solution for the one-dimensional problem in the time domain has not been reported in the literature. Li and Schanz [11] presented an analytical solution for a single-layer problem in the Laplace domain based on the theory of mixture, and the time-domain solution is obtained using the numerical inverse Laplace transform.

The exact solution can illustrate the transient response of unsaturated porous media, as well as be used to evaluate the accuracy of various numerical results. Moreover, clear insight into the one-dimensional transient response is necessary for understanding wave propagation in unsaturated porous media. Therefore, exact time-domain solutions for the one-dimensional transient response of unsaturated single-layer porous media are of great theoretical and practical importance.

As a companion paper to Shan et al. [10], which is concerned with saturated porous media, this paper develops the exact time-domain solutions for unsaturated single-layer porous media under three types of boundary conditions. In Section 2, we outline the

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http://dx.doi.org/10.1016/j.compgeo.2012.08.001

basic equations for unsaturated porous media, which are proposed by Zienkiewicz et al. [12]. The associated initial conditions and boundary conditions are discussed in Section 3. In Section 4, we present the derivation of the solution in detail. In Section 5, several illustrative examples are given to validate the exact solutions and to illustrate some interesting features of the wave propagation in unsaturated single-layered porous media.

2. Basic equations describing unsaturated porous media

Porous media saturated with two immiscible fluids, i.e., the wetting fluid and the non-wetting fluid, are discussed in this paper. Based on the Biot theory [2], Zienkiewicz et al. [12] and Li et al. [13] developed the basic equations of unsaturated porous media, which are employed to study the transient response of unsaturated porous media. It is assumed that the wave propagates along the z direction in a Cartesian coordinate system and that both the wetting fluid and the non-wetting fluid obey Darcy's Law. The momentum balance equations for the solid and the fluids are:

$$\sigma_{zz,z} - \rho U_{z,tt} - \rho_w W_{z,tt} - \rho_g V_{z,tt} = 0$$

$$P_{w,z} + \rho_w U_{z,tt} + \rho_w W_{z,tt}/(nS_w) + W_{z,t}/k_w = 0$$

$$P_{n,z} + \rho_n U_{z,tt} + \rho_n V_{z,tt}/(nS_n) + V_{z,t}/k_n = 0, \tag{1}$$

where subscripts w and n denote the wetting fluid and the non-wetting fluid, respectively. σ_z is the total stress, and the tension is positive. P_w and P_n are the wetting fluid pressure and the non-wetting fluid pressures, respectively, and the pressure is positive. P is the effective pressure and can be expressed as $P = S_wP_w + S_nP_n$. U_z is the solid displacement. W_z and V_z are the relative displacements of the wetting fluid and the non-wetting fluid to the solid, respectively. They can be expressed as $W_z = nS_w(\overline{W}_z - U_z)$ and $V_z = nS_n(\overline{V}_z - U_z)$, where $\overline{W}_z$ and $\overline{V}_z$ are the absolute displacements of the wetting fluid and the non-wetting fluid, respectively. n is the porosity. S_w and S_n are the saturations of the wetting fluid and the non-wetting fluid, respectively, which satisfy the relation $S_w + S_n = 1$. ρ is the density of the unsaturated porous medium, which can be expressed as $\rho = (1-n)\rho_s + nS_w\rho_w + nS_n\rho_n$, where ρ_s, ρ_w, and ρ_n are the densities of the solid particles, the wetting fluid, and the non-wetting fluid, respectively. k_w and k_n are the dynamic permeability coefficients of the wetting fluid and the non-wetting fluid, respectively. $(\,)_{,z}$ and $(\,)_{,t}$ denote the partial derivatives with respect to z and t, respectively.

Assuming that the wetting fluid and the non-wetting fluid can flow independently, the flow conservation equations for the wetting fluid and the non-wetting fluid can be written as:

$$Q_wW_{z,z} + \alpha Q_wS_wU_{z,z} + S_wP_w = 0, Q_nV_{z,z} + \alpha Q_nS_nU_{z,z} + S_nP_n = 0, \tag{2}$$

where α, Q_w, and Q_n are parameters that describe the compressibility of the unsaturated porous medium. These three parameters can be expressed as $\alpha = 1 - K_b/K_s, 1/Q_w = (\alpha - n)/K_s + n/K_w$, and $1/Q_n = (\alpha - n)/K_s + n/K_n$, where K_b, K_s, K_w, and K_n are the bulk moduli of the solid skeleton, the solid grains, the wetting fluid, and the non-wetting fluid, respectively. The bulk modulus of the skeleton can be described by $K_b = \lambda + 2\mu/3$, where λ and μ are the Lamé parameters of the solid skeleton.

The constitutive equation of the solid skeleton is:

$$\sigma_{zz} + \alpha(S_wP_w + S_nP_n) = (\lambda + 2\mu)U_{z,z}. \tag{3}$$

It should be noted that the saturations S_w and S_n , the porosity n, and other material parameters in Eqs. (1)–(3) are considered to be constants.By introducing the following dimensionless variables:

$$\zeta = \frac{z}{H}, \quad u = \frac{U_z}{H}, \quad w = \frac{W_z}{H}, \quad v = \frac{V_z}{H}, \quad p_n = \frac{P_n}{m}, \quad p_w = \frac{P_w}{m}$$

$$\sigma = \frac{\sigma_{zz}}{m}, \quad \tau = \frac{\sqrt{m/\rho}}{H}t, \quad m = \lambda + 2\mu + \alpha^2Q_wS_w + \alpha^2Q_nS_n, \tag{4}$$

and the following dimensionless coefficients:

$$\beta_w = \rho_w/\rho, \beta_n = \rho_n/\rho, \gamma_w = \beta_w/nS_w), \gamma_n = \beta_n(nS_n)$$

$$\eta_w = H/(2k_w\sqrt{m\rho}), \kappa_w = Q_w/m, \eta_n = H/(2k_n\sqrt{m\rho}), \kappa_n = Q_n/m, \tag{5}$$

where H is the single-layer thickness, Eqs. (1)–(3) can be made dimensionless. They can then be written compactly in matrix form:

$$\boldsymbol{\sigma}_{,\zeta} - \boldsymbol{M}\boldsymbol{u}_{,\tau\tau} - \boldsymbol{C}\boldsymbol{u}_{,\tau} = \boldsymbol{0} \tag{6a}$$

$$\boldsymbol{\sigma} = \boldsymbol{K}\boldsymbol{u}_{,\zeta}, \tag{6b}$$

where

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma \\ -p_w \\ -p_n \end{Bmatrix}, \quad \boldsymbol{u} = \begin{Bmatrix} u \\ w \\ v \end{Bmatrix}, \quad \boldsymbol{K} = \begin{bmatrix} 1 & \alpha\kappa_w & \alpha\kappa_n \\ \alpha\kappa_w & \kappa_w/S_w & 0 \\ \alpha\kappa_n & 0 & \kappa_n/S_n \end{bmatrix},$$

$$\boldsymbol{M} = \begin{bmatrix} 1 & \beta_w & \beta_n \\ \beta_w & \gamma_w & 0 \\ \beta_n & 0 & \gamma_n \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\eta_w & 0 \\ 0 & 0 & 2\eta_n \end{bmatrix} \tag{7}$$

Substituting Eq. (6b) into (6a), the governing equation in terms of $\boldsymbol{u}$ is:

$$\boldsymbol{K}\boldsymbol{u}_{,\zeta\zeta} - \boldsymbol{M}\boldsymbol{u}_{,\tau\tau} - \boldsymbol{C}\boldsymbol{u}_{,\tau} = \boldsymbol{0}. \tag{8}$$

3. Initial conditions and boundary conditions

To address the transient response of an unsaturated single-layer porous medium, as shown in Fig. 1, associated initial conditions and boundary conditions must be imposed. In this paper, the following initial conditions are employed:

$$\boldsymbol{u}(\zeta, 0) = \boldsymbol{g}_1(\zeta), \boldsymbol{u}_{,\tau}(\zeta, 0) = \boldsymbol{g}_2(\zeta), \tag{9}$$

where $\boldsymbol{g}_1(\zeta) = \{g_{11}(\zeta),\ g_{12}(\zeta),\ g_{13}(\zeta)\}^T$ and $\boldsymbol{g}_2(\zeta) = \{g_{21}(\zeta),\ g_{22}(\zeta),\ g_{23}(\zeta)\}^T$, with $g_{ij}(\zeta)$ being specified functions in various forms.

The following three types of nonhomogeneous boundary conditions are also considered:

$$\text{(A)} \quad \boldsymbol{u}(0, \tau) = \boldsymbol{f}_1(\tau), \boldsymbol{u}(1, \tau) = \boldsymbol{f}_2(\tau) \tag{10a}$$

$$\text{(B)} \quad \boldsymbol{u}_{,\zeta}(0, \tau) = \boldsymbol{f}_1(\tau)\boldsymbol{u}(1, \tau) = \boldsymbol{f}_2(\tau) \tag{10b}$$

$$\text{(C)} \quad \boldsymbol{u}_{,\zeta}(0, \tau) = \boldsymbol{f}_1(\tau), \boldsymbol{u}_{,\zeta}(1, \tau) = \boldsymbol{f}_2(\tau), \tag{10c}$$

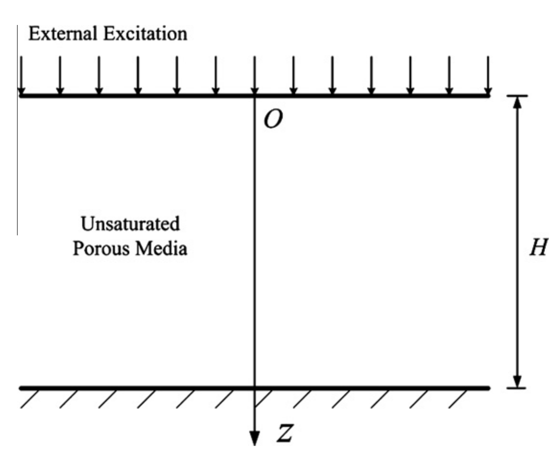

Fig. 1. The unsaturated porous medium.