Computers and Geotechnics 48 (2013) 260–271

Contents lists available at SciVerse ScienceDirect

Computers and Geotechnics

journal homepage: www.elsevier.com/locate/compgeo

A perturbation method for optimization of rigid block mechanisms in the kinematic method of limit analysis

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A R T I C L E   I N F O

Article history:
Received 16 February 2012
Received in revised form 27 July 2012
Accepted 27 July 2012
Available online 13 October 2012

Keywords:
Kinematic method
Upper bound
Limit analysis
Rigid block
Optimization
Perturbation
Cone programming

A B S T R A C T

Collapse mechanisms consisting of sliding rigid blocks are used widely as the basis for computing bounds on limit loads in geotechnical and structural engineering problems. While these mechanisms are conceptually straightforward to analyze, evaluating kinematically admissible velocities for a particular arrangement of blocks can be a tedious process, and optimizing the geometry of the mechanism is often prohibitively cumbersome for more than a few blocks. In this paper, we present a numerical technique for evaluating and optimizing mechanisms composed of an arbitrary number of sliding triangular blocks, assuming plane strain and homogenous, ponderable material obeying the Mohr–Coulomb yield condition. In the proposed method, coordinates defining the vertices of the blocks are treated as unknowns, and the optimal geometry is found by successively perturbing the vertex coordinates and block velocities, starting initially from a user-specified arrangement of blocks. The method is applied to three different examples related to geotechnical engineering, each of which illustrate that the approach is an efficient way to evaluate bounds that are often close to the true limit load.

1. Introduction

The kinematic method of limit analysis is a well-established technique for evaluating bounds on limit loads for engineering structures. As discussed in detail by Chen [1], the method pertains to materials that can be idealized as perfectly plastic with associated plastic flow, and it rests on constructing a kinematically admissible velocity field (i.e., collapse mechanism) in which strain rates everywhere satisfy the plastic flow rule and velocities satisfy boundary conditions. For any kinematically admissible mechanism, the limit load computed by balancing the rate of dissipation by plastic deformation to the rate of work done by external forces is a rigorous bound on the true limit load. The bound is an upper bound for loads inducing collapse and a lower bound for loads resisting collapse. Regrettably, the kinematic method has come to be known more commonly as "upper bound limit analysis," which in many instances belies the nature of the analysis and the computed bound.

The kinematic method is most often implemented in one of two forms. The first, here referred to as the "analytical method," relies on postulating and geometrically constructing an admissible collapse mechanism (e.g., [2–10]). This approach typically furnishes a closed-form expression of the limit load that includes unknown parameters characterizing the geometry of the mechanism, and the objective is then to assess the values for which the computed limit load is optimal, thereby bracketing the true limit load as closely as possible. For simplicity, it is common to assume a mechanism consisting of sliding or rotating rigid blocks separated by transition layers of infinitesimal thickness across which the velocity is discontinuous (i.e., velocity discontinuities). In such a "rigid block mechanism," the interior angles of the blocks represent the unknown geometric parameters to be optimized. When the collapse mechanism consists of only a few rigid blocks or follows a special pattern, it is possible to evaluate the optimal geometry analytically or using straightforward numerical techniques. In general, however, the analytical method is impracticable when the number of blocks is large.

The second, purely numerical form of limit analysis is so-called finite element limit analysis (FELA). The central idea of FELA is to subdivide the entire problem domain into a number of elements over which the complete velocity field is obtained by interpolating discrete values at nodal points (e.g., [11–18]). By assigning each element a unique set of nodes, velocity discontinuities corresponding to element edges are also permitted. Whereas the velocity field in the analytical method is kinematically admissible by construction, admissibility in FELA is ensured only by introducing a large set of constraints on the otherwise arbitrary velocity field as part of the procedure for optimizing the limit load. Unlike the analytical method, the constrained optimization problem emerging in FELA

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0266-352X/$ - see front matter
http://dx.doi.org/10.1016/j.compgeo.2012.07.012

can be solved using large-scale mathematical programming (e.g., linear or nonlinear programming), allowing for a virtually arbitrary number of elements. When velocities are assumed constant within elements, the velocity field in FELA reverts to a rigid block mechanism, and several studies have demonstrated the effectiveness of such an approach [19–22].

Even when continuous deformation is permitted in the collapse mechanism, the majority of plastic dissipation typically occurs along velocity discontinuities, and therefore determining their optimal locations is essential. In the analytical method, the optimization process is tantamount to finding these locations. However, the accuracy of this method is often limited by the number of blocks that can be practically analyzed, and a generally accepted method for optimizing mechanisms with a large number of blocks in an arbitrary arrangement has yet to be established. In FELA, the need to define the placement of (potential) velocity discontinuities *a priori* is a basic shortcoming of the approach, as it is unlikely that the element edges will coincide with the optimal locations. The leading approach for overcoming this drawback is to use anisotropic mesh adaptivity, which locally refines the finite element mesh based on estimates of the velocity gradient and the Hessian matrix [23]. Resolving the optimal locations of velocity discontinuities using mesh adaptivity therefore relies on introducing a potentially large number of additional elements through the course of multiple iterations.

Two different approaches for directly optimizing the locations of velocity discontinuities in a collapse mechanism can be found in the literature, and both pertain to rigid block mechanisms. The first is known as discontinuity layout optimization (DLO) [24], and it is based on identifying the optimal connectivity of potential velocity discontinuities spanning a fixed grid of nodes [24]. The second approach rests on first evaluating an admissible collapse mechanism using methods similar to those utilized in FELA and then optimizing its geometry by a series of small adjustments, or perturbations, using sequential linear programming [25], a concept first introduced by Johnson [26] for rigid-plastic analysis of concrete slabs. While both DLO and the approach based on sequential linear programming have proven to be effective in analyzing stability problems, an advantage of the latter is that it can resolve the optimal locations of velocity discontinuities anywhere in space. In DLO, velocity discontinuities must always span two nodes, and the resolution therefore depends on the grid spacing.

This paper expands on the method introduced by Milani and Lourenço [25] and presents a new computational approach for optimizing rigid block mechanisms consisting of an arbitrary number of sliding triangular blocks. The approach utilizes large-scale mathematical programming firstly to compute admissible velocities for an initial, user-specified arrangement of blocks and then to optimize the geometry of the mechanism through a sequence of successive perturbations. As compared to the method described by Milani and Lourenço [25], key enhancements are as follows:

1. Within each perturbation step, the optimization problem is cast concisely as a second-order cone programming problem, rather than a linear programming problem.
2. The formulation includes a simple means for ensuring that the mechanism remains valid within each perturbation step.
3. A strategy for progressively adjusting the magnitude of the perturbations and a corresponding stopping criterion are proposed.
4. Material self-weight, which is essential for geotechnical problems, is included.

Milani and Lourenço [25] also considered rotating blocks whose edges are defined by Bézier curves, although such a mechanism is generally inadmissible and does not furnish a rigorous bound. Here it is assumed that the blocks possess straight edges and do not rotate.

The proposed formulation pertains to plane strain and material obeying the Mohr–Coulomb yield condition and, as a matter of convenience, it is assumed that the limit load induces, rather than resists, collapse. Therefore, all computed bounds are upper bounds. Extension to the case of lower bounds on loads resisting collapse is straightforward. A rudimentary version of the proposed approach, without features (2)–(4) above, was presented in an earlier conference paper by the authors [27].

In the next section, an approach for computing admissible velocities for a predefined arrangement of sliding rigid blocks is presented. In Section 3, the formulation for a fixed arrangement of blocks is adapted so that the mechanism geometry is optimized through successive perturbation. Section 4 is devoted to examples, and the penultimate section presents observations about the approach and possible areas of future research.

2. Rigid block mechanism of fixed geometry

The starting point for the formulation is to subdivide the problem domain into a number of contiguous triangular blocks (elements). As a convention, each vertex (node) in the assembly is identified by index i, where $i = 1, 2, \ldots, N_V$, and each block is identified by index j, where $j = 1, 2, \ldots, N_B$. The Cartesian coordinates of vertex i are denoted by x_i and y_i, and components of velocity in block j are $v_{x,j}$ and $v_{y,j}$.

With the exception of edges on free boundaries, each edge in the arrangement of blocks represents a potential velocity discontinuity. Velocity discontinuities are denoted by index k, where $k = 1, 2, \ldots, N_E$, and the local quantities associated with a particular discontinuity are as shown in Fig. 1. The vertices S and N are the "south" vertex and "north" vertices, respectively, and the blocks on either side of the discontinuity are similarly denoted by E and W, which indicate the "east" and "west" blocks. Using this convention, which is independent of the discontinuity's orientation, the following vectors containing local quantities for discontinuity k are defined

$$\mathbf{x}_k = [x_S \ y_S \ x_N \ y_N]^T, \quad \mathbf{v}_k = [v_{x,E} \ v_{y,E} \ v_{x,W} \ v_{y,W}]^T \tag{1}$$

In Eq. (1), x and y are the coordinates of the local vertices, with the subscript indicating the vertex, and v_x and v_y are the components of velocity in adjacent blocks, with the subscript indicating the block. When material on one side of the discontinuity is at rest, components of velocity on that side are simply taken as zero.

For kinematic admissibility, the following jump condition must be imposed at each velocity discontinuity (see, for instance [1])

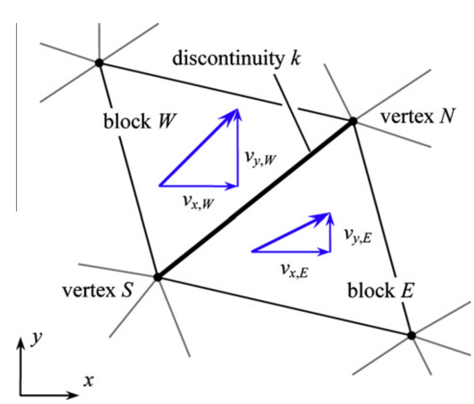

Fig. 1. Local vertex and block designations for a velocity discontinuity between two adjacent blocks.