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Prolegomena to variational inequalities and numerical schemes for compressible viscoplastic fluids☆

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ABSTRACT

Firstly, a summary of the development of the constitutive equation for an incompressible Bingham fluid, the variational inequality and an operator-splitting numerical method for the solution of isothermal flow problems is presented. Next, a variational inequality is derived for compressible viscoplastic fluids in which the viscosity and yield stress depend on the pressure, temperature and the three invariants of the first Rivlin–Ericksen tensor, and inertial effects are present. Based on a known version for compressible Newtonian fluids, an extension of the operator-splitting scheme to the flows of compressible viscoplastic fluids, when inertia and thermal effects are manifest, is proposed. Finally, this scheme is employed to examine the isothermal flow of a Bingham material in a square cavity when the fluid is slightly compressible, with its density depending linearly or exponentially on the pressure. The results are compared with those for an incompressible fluid for small Bingham numbers.

1. Introduction

More than 30 years ago, Duvaut and Lions [1] demonstrated that variational inequalities (VIs) arise naturally in the solution of steady and unsteady flow problems in incompressible Bingham fluids. As is well known, the flow domain of a Bingham fluid consists of parts where the fluid has yielded and where it is at rest, or moves as a rigid body. Clearly, any method used to solve a flow problem has to identify these zones. The discovery that Bingham fluid flows obey VIs was extremely important because during the solution of a VI, the yielded and unyielded zones in a given flow problem manifest themselves without any a priori assumptions regarding their locations or size. Subsequently using the relevant VI, Glowinski [2] showed that the flow of a Bingham fluid in a pipe of arbitrary cross-section comes to a halt in a finite amount of time, if the applied pressure gradient drops below a geometrically defined multiple of the yield stress; in fact, a finite upper bound to this extinction time was found in Ref. [2]. This fundamental difference between a viscous fluid, such as a Newtonian fluid, and a Bingham fluid was indeed surprising and has been shown to exist between the Newtonian and other viscoplastic fluids as well; see [3].

In their book, Duvaut and Lions [1] proved further that the VI was equivalent to the existence of a symmetric second order tensor field throughout the flow domain. This tensor, called a *multiplier*, is such that its 'magnitude' is less than one where the Bingham fluid exhibits rigidity and equal to one where it has yielded. Moreover, the inner product of this tensor with the first Rivlin–Ericksen tensor [4], associated to the velocity field which satisfies the equations of motion, leaves the magnitude of the latter unaffected. This was exploited by Cea and Glowinski [5] brilliantly to develop numerical methods based on an Uzawa type algorithm to model the steady flows of Bingham fluids in pipes of arbitrary cross-section. More recently, the multiplier has been used in connection with operator-splitting methods to model other flows of Bingham fluids numerically [6–9]. Our aim is to suggest an operator-splitting method for the flows of compressible yield stress fluids based on the work of Li and Glowinski [9,10].

In order to fulfil this task, a review of the operator-splitting method for incompressible Bingham fluids has to be provided first. In turn, this means that one has to look carefully at the procedure for deriving the constitutive equation for a Bingham fluid, which has to be prefaced by an examination of incompressibility in fluid mechanics in general, and the response of the Bingham fluid to the additional constraint of yielded-unyielded regions arising from the bounds on the 'magnitude' of the extra stress tensor. These matters are explored in Sections 2 and 3 and the multiplier is introduced into the constitutive equation through the VI, accompanied by a caveat.

In order to solve the VI for complicated flows, numerical methods have to be discovered. One of them is the operator-splitting

☆ Sections 1–7 were presented as part of the plenary lecture by R.R.H. at the Monte Verità meeting. Section 8, consisting of the application of the operator-splitting method to flows in a cavity by Z.Y., is added here.

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method for Bingham fluids due to Sánchez [6]. A similar scheme has also been proposed by Dean et al. [7]. In essence, this method divides the task of finding the velocity, the multiplier tensor and pressure fields into three sub-problems and iteration proceeds till convergence occurs. These matters are discussed in Section 4, and applied in Section 8 to the flow in a square cavity confirming the earlier results [8,9].

To develop such a scheme for compressible viscoplastic fluids, the VI for such a fluid is derived ab initio in Section 5 and the continuity and energy equations are listed in Section 6. Finally in Section 7, we look briefly at the isentropic flows of compressible viscous fluids [11,12] and find that the perturbation method [9] is difficult to apply to compressible viscoplastic fluids. This is because the perturbation of the velocity field may change the location and shape of the yielded-unyielded regions. Thus, iteration beyond the basic flow is not viable numerically.

Turning to the main goal of Section 7, we summarise the operator-splitting method of Li and Glowinski [9,10] for low Mach number flows of compressible viscous fluids when thermal and inertial effects are present. Subsequently, this method is extended to compressible viscoplastic fluids in the same spirit as that for incompressible fluids mentioned in Section 4. As an application, the flow of a slightly compressible Bingham fluid in a square cavity is examined in Section 8, when the density depends linearly or exponentially on the pressure. The results are compared with those for an incompressible Bingham fluid for small Bingham numbers.

2. Incompressible fluids

We begin by writing the constitutive equation of an incompressible viscoplastic fluid as

$$\mathbf{S} = -p\mathbf{1} + \boldsymbol{\tau}, \tag{1}$$

where p is the pressure and $\boldsymbol{\tau}$ is the extra stress tensor. In this work, we shall use the first[1] Rivlin–Ericksen tensor [4]. $\mathbf{A}$, rather than the rate of deformation tensor $\mathbf{D}$.

The first Rivlin–Ericksen tensor $\mathbf{A}$ is derived from a velocity field $\mathbf{v}$ as follows:

$$\mathbf{A}(\mathbf{v}) = \nabla\mathbf{v} + \nabla\mathbf{v}^{\mathrm{T}} = 2\mathbf{D}. \tag{2}$$

One of the invariants which we shall use is the trace of $\mathbf{A}$, i.e., $\mathrm{tr}\mathbf{A} = I_1(\mathbf{A}) = \mathbf{1} : \mathbf{A}$.

Secondly, we shall call upon another invariant $K \geq 0$, which is defined below:

$$2K^2(\mathbf{v}) = \mathbf{A}(\mathbf{v}) : \mathbf{A}(\mathbf{v}). \tag{3}$$

The reason is that in a simple shearing flow given by $\mathbf{v} = \dot{\gamma} y \mathbf{i}$, where the shear rate $\dot{\gamma} > 0$ is a constant, we get

$$2K^2(\mathbf{v}) = 2\dot{\gamma}^2, \tag{4}$$

so that $K(\mathbf{v}) = \dot{\gamma}$, the magnitude of the shear rate. We shall also use the invariant $2K^2(\boldsymbol{\tau}) = \boldsymbol{\tau} : \boldsymbol{\tau}$ of the extra stress tensor.

The first task is to describe the VI for an incompressible viscoplastic fluid and study some of its consequences. However, before we get to this inequality, we have to answer the following questions:

(1) What is meant by the pressure p in an incompressible fluid?
(2) Can the material properties such as the viscosity η, and the yield stress τ_y, depend on p?
(3) How can we apply these ideas to incompressible viscoplastic fluids?

First of all, we know that the constitutive equation in an incompressible fluid does not define the pressure term p. The latter is said to arise in an incompressible material as a response to the constraint imposed on the material, which is $\mathbf{1} : \mathbf{A} = 0$ in all motions. Is there a way of defining p so that it is unique? Note that in viscoelastic fluids, this point is ignored because the material properties such as viscosity, the normal stress differences, the dynamic moduli and the extensional viscosity are the same whether we define p through $p = -(1/3)\mathbf{1} : \mathbf{S}$ or not.

In viscoplastic fluids, we do not have this luxury because the existence of a rigid flow domain depends on the invariant $K(\boldsymbol{\tau})$ of the extra stress tensor. So, the decomposition of the total stress $\mathbf{S}$ into $-p\mathbf{1}$ and $\boldsymbol{\tau}$ is critical.

Recently, Rajagopal and Srinivasa [13] have investigated the response in a continuous medium subject to a single scalar constraint. Suppose that the constraint is defined through an equation of the form $\phi(\mathbf{A}) = 0$. If this represents a 'surface', one can suppose that it has a normal $\mathbf{N}$, which is a symmetric second order tensor defined through

$$\mathbf{N} = \frac{\partial\phi}{\partial\mathbf{A}}. \tag{5}$$

Thus, the total stress tensor $\mathbf{S}$ is decomposed as:

$$\mathbf{S} = \lambda\mathbf{N} + \boldsymbol{\tau}, \qquad \mathbf{N} : \boldsymbol{\tau} = 0, \tag{6}$$

so that

$$\lambda = \frac{\mathbf{S} : \mathbf{N}}{\mathbf{N} : \mathbf{N}}. \tag{7}$$

In defining λ in this manner, one is not invoking any argument to the effect that the 'constraint forces' do no virtual work in a constrained motion.

Now, the decomposition of $\mathbf{S}$ into $\lambda\mathbf{N}$ and $\boldsymbol{\tau}$ does not preclude the latter from depending on λ. In fact, if one looks at Coulomb friction problems in rigid body mechanics where sliding motion is present, the friction force opposing the motion is proportional to the constraint force provided by the surface on which the motion occurs.

Turning to incompressible fluids, one can show quite easily that $\mathbf{N} = \mathbf{1}$, the identity tensor. Then, we have the unique decomposition

$$\mathbf{S} = -p\mathbf{1} + \boldsymbol{\tau}, \qquad \mathbf{1} : \boldsymbol{\tau} = 0. \tag{8}$$

Thus, $p = -(1/3)\mathbf{1} : \mathbf{S}$ is uniquely defined, and $\boldsymbol{\tau}$ can depend on p as well. It follows from this result that in the yielded zone in a viscoplastic fluid, one has a unique definition of p, because in such a fluid $\boldsymbol{\tau} = f(K(\mathbf{v}))\mathbf{A}(\mathbf{v})$.

More than this is required. In the rigid zone, p cannot be defined by the above procedure because the constraint now is $\mathbf{A} = \mathbf{0}$, which is more restrictive than $\mathbf{1} : \mathbf{A} = 0$. In addition, we must ensure that the definition of pressure in the rigid zone leads to an entity which is continuous with that in the yielded region. So, we define $p = -(1/3)\mathbf{1} : \mathbf{S}$ throughout the flow domain Ω. Then, the constraints on the second invariant of $\boldsymbol{\tau}$ in the rigid and yielded zones are uniquely defined. If $p = -(1/3)\mathbf{1} : \mathbf{S}$, the pressure term is well defined even if a fluid particle moves into a sheared region and out of it as discussed by Frigaard and Ryan [14].

Another difficulty with the constitutive equation of a viscoplastic fluid is that the demarcation of the rigid zone from the yielded zone is really a constraint, and it would seem that a separate restriction on the constitutive relation must arise. We shall examine this aspect next.

[1] The usual notation for the first Rivlin–Ericksen tensor [4] is $\mathbf{A}_1$. Since we do not need the higher order Rivlin–Ericksen tensors, we shall denote the first tensor without the subscript, i.e., as $\mathbf{A}$.