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Re-entrant corner flow for PTT fluids in the natural stress basis

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ABSTRACT

We revisit the situation of steady planar flow of Phan–Thien–Tanner (PTT) fluids around re-entrant corners of angles π/α where $1/2 \le \alpha < 1$. The model is considered in the absence of a solvent viscosity, under which a class of self-similar solutions has been identified with stress singularities of $O(r^{-2(1-\alpha)})$ and stream function behaviour $O(r^{\alpha(1+\alpha)})$ (r being the radial distance from the corner). The asymptotic analysis is completed by providing a solution for the downstream boundary layer using natural stress variables. We show that the matching of the outer (core) solution into the downstream boundary layer imposes a restriction on the range of $\alpha \in (2/3, 1)$ for which these self-similar solutions are applicable, i.e. they only hold for corner angles between 180° and 270°.

1. Introduction

In [4], a local asymptotic behaviour was identified for the steady planar flow of Phan–Thien–Tanner (PTT) fluids at re-entrant corners. The PTT model was considered in the absence of a solvent viscosity with complete flow assumed around the corner. A familiar three region asymptotic structure was determined comprising an outer region away from the walls together with elastic wall boundary layers upstream and downstream. This structure parallels that for the UCM model [2]. An analogous class of self-similar solutions was derived for the outer region sharing the same stress singularity as for the UCM model but differing in the stream function behaviour (this latter effect being attributed to PTT's shear-thinning properties). A solution for the upstream boundary layer was obtained but not for the downstream layer, and it is the latter case that we wish to address here. Similar to UCM [3](and Oldroyd-B [9]), the use of the natural stress basis facilitates the problem formulation, not least in identifying the information that is communicated from the upstream boundary layer through the outer region and subsequently into the downstream layer.

We refer to [4] for the setup and governing equations, with Fig. 1 summarising the leading order governing equations in the main asymptotic regions. Usual polar coordinates (r, θ) are taken centered at the corner with the upstream wall being $\theta = 0$ and the downstream wall $\theta = \pi/\alpha$ with $1/2 \le \alpha < 1$. In the rest of the Introduction we record the governing equations in the natural stress basis, with the outer region addressed in Section 2 and the stress boundary layers considered in Section 3. We show that a downstream solution can only be constructed for corner angles strictly between 180° and 270°. The downstream solution for corner angles in the range [270°, 360°] remains outstanding. The presence of a solvent viscosity significantly alters the results presented here, further remarks on which are made in Section 6.

Initially, Cartesian axes are taken as shown in Fig. 1 with the velocity **v** and extra-stress **T** fields expressed in the form

$$\mathbf{v} = (u, v)^T = \left(\frac{\partial\psi}{\partial x}, -\frac{\partial\psi}{\partial y}\right)^T, \quad \mathbf{T} = -\mathbf{I} + \lambda\mathbf{v}\mathbf{v}^T + \mu(\mathbf{v}\mathbf{w}^T + \mathbf{w}\mathbf{v}^T) + \nu\mathbf{w}\mathbf{w}^T, \tag{1.1}$$

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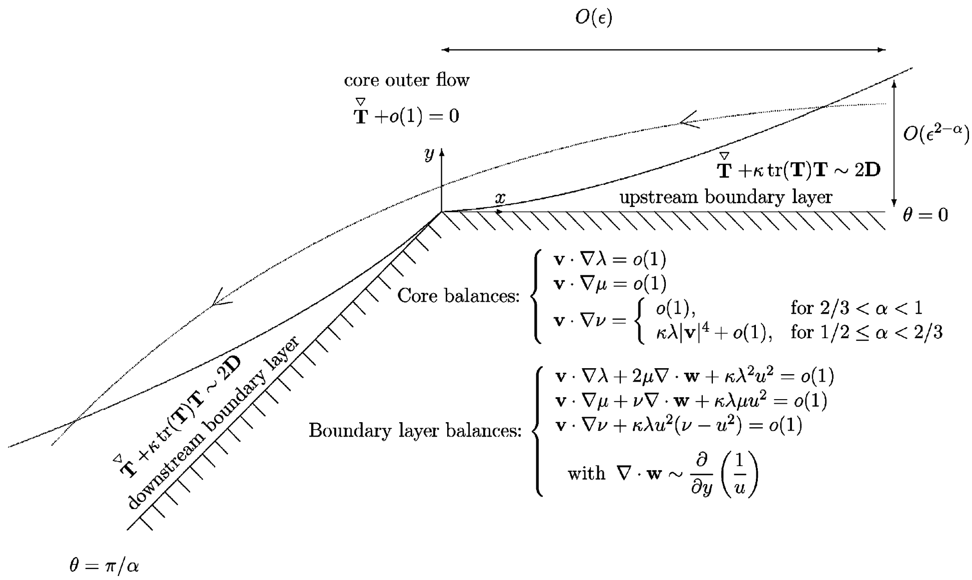

Fig. 1. Schematic illustration of the asymptotic regions local to the re-entrant corner. Shown is the three region structure, comprising the outer (core flow) region away from the boundaries and inner regions (stress boundary layers) at the upstream and downstream walls. The leading order balances are shown for the respective regions, the boundary layer equations being those associated with high Weissenberg number limit [7]. This structure holds on small length scales of $O(\epsilon)$, ϵ being a small artificial parameter, the boundary layers are of thickness $O(\epsilon^{2-\alpha})$.

where $\psi(x, y)$ is the usual stream function, $\lambda(x, y)$, $\mu(x, y)$, and $\nu(x, y)$ are the natural stress variables and $\mathbf{w}$ is a vector orthogonal to $\mathbf{v}$ given by

$$\mathbf{w} = (w_1, w_2)^T = \left(-\frac{\nu}{u^2+\nu^2}, \frac{u}{u^2+\nu^2}\right)^T.$$

The dimensionless momentum and constitutive equations in component form can be expressed as

$$0 = -\frac{\partial p}{\partial x} + (\mathbf{v}\cdot\nabla)(\lambda u) + \nabla\cdot(\mu u\mathbf{w} + (\mu\mathbf{v} + \nu\mathbf{w})w_1), \tag{1.2}$$

$$0 = -\frac{\partial p}{\partial y} + (\mathbf{v}\cdot\nabla)(\lambda \nu) + \nabla\cdot(\mu \nu\mathbf{w} + (\mu\mathbf{v} + \nu\mathbf{w})w_2), \tag{1.3}$$

and

$$\lambda + (\mathbf{v}\cdot\nabla)\lambda + 2\mu\nabla\cdot\mathbf{w} + \kappa\left(\lambda|\mathbf{v}|^2 - 2 + \frac{\nu}{|\mathbf{v}|^2}\right)\left(\lambda - \frac{1}{|\mathbf{v}|^2}\right) = \frac{1}{|\mathbf{v}|^2}, \tag{1.4}$$

$$\mu + (\mathbf{v}\cdot\nabla)\mu + \nu\nabla\cdot\mathbf{w} + \kappa\left(\lambda|\mathbf{v}|^2 - 2 + \frac{\nu}{|\mathbf{v}|^2}\right)\mu = 0, \tag{1.5}$$

$$\nu + (\mathbf{v}\cdot\nabla)\nu + \kappa\left(\lambda|\mathbf{v}|^2 - 2 + \frac{\nu}{|\mathbf{v}|^2}\right)(\nu - |\mathbf{v}|^2) = |\mathbf{v}|^2, \tag{1.6}$$

where p is the pressure, $\kappa = O(1)$ is a positive dimensionless model parameter (a mobility factor) and

$$\nabla\cdot\mathbf{w} = \frac{1}{(u^2+\nu^2)^2}\left((\nu^2 - u^2)\left(\frac{\partial \nu}{\partial x} + \frac{\partial u}{\partial y}\right) + 4u\nu\frac{\partial u}{\partial x}\right).$$

In the momentum equations, the inertia terms have been neglected (since they have been shown in [4] to be uniformly negligible at leading order in each of the asymptotic regions) and the dimensionless relaxation time (the Weissenberg number) taken as unity in the constitutive equations. These equations are supplemented with no slip conditions on the solid boundaries $\theta = 0$ and $\theta = \pi/\alpha$.

For later scaling purposes, we record the leading order solution in the outer region $r = o(1)$ away from the walls, which was shown in [4] to be

$$\psi = \frac{C_0}{\alpha^n} r^{\alpha n}\sin^n(\alpha\theta), \qquad p = \frac{1}{2}\lambda(\psi)|\mathbf{v}|^2 = p_0 r^{-2(1-\alpha)}, \qquad \mathbf{T} = \lambda(\psi)\mathbf{v}\mathbf{v}^T, \quad \text{as } r \to 0, \tag{1.7}$$

with

$$\lambda(\psi) = \frac{2p_0}{n^2C_0^2}\left(\frac{\psi}{C_0}\right)^{(2/n)-2}.$$