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Short communication

Stress singularities and the formation of birefringent strands in stagnation flows of dilute polymer solutions

Paul Becherer [a,*], Wim van Saarloos [a], Alexander N. Morozov [b]

[a] Instituut-Lorentz for Theoretical Physics, Universiteit Leiden, Postbus 9506, NL-2300 RA Leiden, The Netherlands
[b] School of Physics, University of Edinburgh, JCMB, King's Buildings, Mayfield Road, Edinburgh EH9 3JZ, United Kingdom

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ABSTRACT

We consider stagnation point flow away from a wall for creeping flow of dilute polymer solutions. For a simplified flow geometry, we explicitly show that a narrow region of strong polymer extension (a birefringent strand) forms downstream of the stagnation point in the UCM model and extensions, like the FENE-P model. These strands are associated with the existence of an essential singularity in the stresses, which is induced by the fact that the stagnation point makes the convective term in the constitutive equation into a singular point. We argue that the mechanism is quite general, so that all flows that have a separatrix going away from the stagnation point exhibit some singular behaviour. These findings are the counterpart for wall stagnation points of the recently discovered singular behaviour in purely elongational flows: the underlying mechanism is the same while the different nature of the singular stress behaviour reflects the different form of the velocity expansion close to the stagnation point.

1. Introduction

Extensional flows of polymer solutions and melts occur in many industrial polymer processing operations, and hence such flows have been studied for decades [1,2]. Recently however, interest in extensional flows was renewed by observations of steady and unstable continuous flow in microfluidic devices [3–5], and it was realized only recently that extensional flows are prone to the formation of singularities and non-analytic structures in the stress fields [6–8]. Depending on the Deborah number and the model used, these stress singularities may take various forms. For purely extensional flow in continuum models that describe infinitely extensible polymer chains (such as the upper convected Maxwell model (UCM) and the Oldroyd-B model [1,2,9]) the stresses can have power law spatial behaviour with a finite limit at the centre line, or they even have power law divergencies. For models that are based on finitely extensible chains, divergencies are cut off at some scale, but singular behaviour of the stress *gradients* may persist [6–8]. Such singular behaviour may have important implications for numerical simulations of extensional flows, since it leads to structures with a very small length scale. Indeed it is known that for many such flows, numerical schemes break down at only moderate flow rates (Deborah numbers of order unity).

The question quite naturally comes up whether singular behaviour near special points is the rule rather than the exception. We argue in this Communication that the latter is the case and demonstrate this for a simplified case where all calculations can be done analytically, so that the emergence of the singular behaviour can be followed explicitly.

The reason to expect singular behaviour near special points where the velocity vanishes – even though the geometry is not singular[1] – is actually very simple. For steady flow, the only derivative terms of the stress $\mathbf{T}$ in UCM-type constitutive equations come from the convective term $(\mathbf{v}\cdot\nabla)\mathbf{T}$. The points where $\mathbf{v}$ vanishes – the stagnation point in elongational flow or in the wall stagnation point flow considered here – thus translate into a singular point [10] of the partial differential equation obtained from the constitutive equation for the stress. Close to the singular point, the lowest order terms in the expansion of $\mathbf{v}$ are often fixed by simple symmetry considerations and boundary conditions, if applicable. So the nature of the dominant singularity at the singular point is generally fixed independent of the precise details of the model. Further away from the singularity, the behaviour will typically depend on the details of the flow profile. All these features are well illustrated by the analysis below. As stated, we focus on a simple case where the calculations can all be done analytically, but the scenario holds generally for

* Corresponding author. Tel.: +31 71 527 5517; fax: +31 71 527 5511.
E-mail address: becherer@lorentz.leidenuniv.nl (P. Becherer).

[1] Of course, at sharp corners where the flow field itself is singular, this singular behaviour carries over to the stresses.

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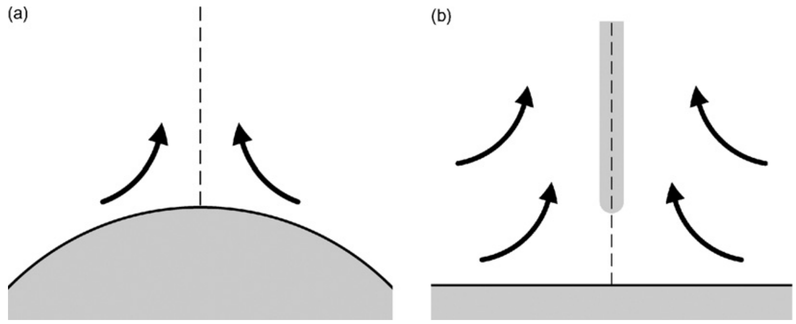

Fig. 1. Stagnation flow (a) in a wake, (b) approximated by a flow near a flat wall. In (b) the formation of a birefringent strand is qualitatively indicated by the shaded area.

complex more realistic flows and we suspect this mechanism of advection to be at the origin of the formation of birefringent strands.

We focus on wall stagnation point flows where the flow is away from the wall. Examples of this are flows in the wake of a falling sphere or of a fixed cylinder, as shown schematically in Fig. 1(a). In particular, the flow past a fixed cylinder or sphere in a channel has become a benchmark problem for numerical modelling of viscoelastic constitutive equations [11–13]. It is known that in such flows a narrow region of high polymer extension may form, a so-called *b* irefringent strand [14,15]. This region starts at a small but finite distance downstream from the stagnation point, as indicated schematically in Fig. 1(b), where a flow near a flat wall is depicted.

In this work, we consider a strictly two-dimensional version of this flow, with a simplified, fixed velocity field obeying the basic symmetry of a stagnation point at a wall (cf. [11]). We analyse this case in detail for the UCM model [1,2,9] but also discuss in the end the qualitative changes that occur for a FENE-P model.

Unlike the case of steady purely extensional flow, which was analysed previously [6–8], the extension of the polymers does not diverge for any extension rate. We find that a thin birefringent strand forms, with a singularity at its centre. As argued above, notwithstanding the simplifications we make in obtaining this result, we believe that the analysis makes it clear how singular behaviour emerges in general.

Unfortunately, our results cannot immediately be compared quantitatively with experiments or numerical computations on realistic cases like flow past a cylinder [12,13]. First, one should keep in mind that in such situations, there may be two sources of (near) singular behaviour: besides the one we analysed here, dominated by the symmetry and boundary conditions of the velocity field near the wall stagnation point, in viscoelastic flow past a cylinder large stress fields are already built up at the sides of the cylinder, where the flow is mostly along the cylinder. These shear stresses are advected toward the rear stagnation point. This effect is clearly not present in the simplified geometry that we consider. Second, our analysis is based on taking a *fixed* velocity field obeying the basic symmetry, and we show how this leads to an essential singularity in the stress. In reality, of course, the velocity and stresses are coupled indirectly through the momentum balance equation. Through this coupling, the velocity field will also be affected in the region of large stress gradients near the stress singularity. Since the symmetry and expansion of the velocity near the stagnation point cannot change in lowest order, we expect that there is an intermediate flow regime where the basic structure of the singularity is not changed dramatically. This assumption is further supported by recent simulations of a two-dimensional cross-slot flow by Poole et al. [5]. There, the velocity profiles remained smooth even when the flow changed its symmetry (the new type of purely elastic instability discovered by Arratia et al. [4]), while stresses exhibit the typical singular structure similar to the one discussed in [6,8]. At the same time, numerical studies suggest that at sufficiently large flow rates, this nonlinear coupling can become so strong that there may be no steady state flow solution past a cylinder for Deborah numbers of order unity [12,16]. The coupling and this effect are, unfortunately, beyond the present approximation.

The layout of this paper is as follows. In Section 2 we introduce the flow geometry and the models, and we briefly recapture similarity solutions for UCM found by other authors [17,18]. We calculate analogous solutions for a simplified version of this flow, where we fix the velocity field, for UCM and FENE-P. In Section 3 we consider more realistic boundary conditions, and we solve the constitutive equations analytically. In Section 4 we consider the resulting stress field (extension field) in more detail, showing that we find a narrow region of high polymer extension, with a non-analytic stress profile at the centre of the strand. We then discuss these results in the light of more realistic flow profiles, and we conclude by discussing the relevance of these results for computational and experimental work.

2. Simplified stagnation flow of a UCM fluid

We consider incompressible planar stagnation flow of a UCM fluid without inertia (creeping flow). The UCM constitutive equation for steady flow is [1]

$$\mathbf{T} + \lambda[(\mathbf{v}\cdot\nabla)\mathbf{T} - (\nabla\mathbf{v})^{\mathrm{T}}\cdot\mathbf{T} - \mathbf{T}\cdot(\nabla\mathbf{v})] = \eta(\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}), \quad (1)$$

where λ is the relaxation time of the fluid and η is the Newtonian viscosity. The momentum balance for creeping flow is

$$\nabla\cdot\mathbf{T} - \nabla p = 0, \quad (2)$$

where p is the pressure. Incompressibility is given by

$$\nabla\cdot\mathbf{v} = 0. \quad (3)$$

The planar stagnation flow geometry we consider is depicted in Fig. 2. We take the vertical direction as bounded, with length ℓ. Because of the solid wall, the boundary condition at the wall ($y = 0$) is $\mathbf{v} = 0$. At $y = \ell$, we impose $v_y = V$, with $V > 0$. For the velocity field, a similarity solution then exists, which is of the form [17,18]

$$v_x = -x\psi'(y) \quad \text{and} \quad v_y = \psi(y), \quad (4)$$

where the boundary conditions imply $\psi(0) = 0$ and $\psi'(0) = 0$. Inspired by this solution, we take a *fixed* velocity field that satisfies the boundary conditions and that would correspond to the lowest-order approximation near the wall:

$$v_x = -2\left(\frac{V}{\ell^2}\right)xy \quad \text{and} \quad v_y = \left(\frac{V}{\ell^2}\right)y^2. \quad (5)$$

Note that these terms are the lowest order analytic terms in an expansion in x and y away from a symmetric stagnation point at the