J. Non-Newtonian Fluid Mech. 155 (2008) 101–115

Journal of Non-Newtonian Fluid Mechanics

www.elsevier.com/locate/jnnfm

An adaptive finite element method for viscoplastic flows in a square pipe with stick–slip at the wall

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Received 23 February 2007; received in revised form 12 November 2007; accepted 3 December 2007

Abstract

This paper presents the numerical solution of non-linear yield stress phenomena by using a new mixed anisotropic auto-adaptive finite element method. The Poiseuille flow of a Bingham fluid with slip yield boundary condition at the wall is considered. Despite its practical interest, for instance for pipeline flows of yield-stress fluids such as concrete and cements, this problem has not been addressed yet to our knowledge. The case of a pipe with a square section has been investigated in detail. The computations cover the full range of the two main dimensionless numbers and exhibit complex flow patterns: all the different flow regimes are completely identified.

Keywords: Viscoplasticity; Bingham fluid; Slip at the wall; Limit load analysis; Variational inequalities; Adaptive mesh; Mixed finite element methods

1. Introduction

The flow of a viscoplastic fluid in a straight pipe with constant cross-section and *with no-slip* condition at the wall has been considered several times in the literature. In 1960s, an extensive mathematical study was presented by Mossolov and Miasnikov [1–3]. These authors have presented impressive results on the existence and shape of rigid zones in the flow. In particular, they were the first to characterize the critical value of the yield stress above which the flow stops. See also Huilgol [4] for a recent application of this approach to several pipe shapes with symmetric cross-section. Next, Duvaut and Lions [5] clarified the the problem of existence and uniqueness of a solution and renewed the mathematical study by using the powerful tools of variational inequalities. They recovered some properties already established by Mossolov and Miasnikov, and found new interesting properties.

The numerical study of this flow problem was first considered in 1972 by Fortin [6]. More recently, the regularized model of Bercovier and Engelman [7] has been used by Taylor and Wilson [8] to study the case of a square cross-section. The augmented Lagrangian algorithm from Fortin and Glowinski [9] has been used by Huilgol and Panizza [10] to solve the case of an annulus and of an L-shaped cross-section, with the Bingham rheology. More recently, Huilgol and You [11] have derived the algorithm for two other viscoplastic rheologies (Casson and Herschel-Bulkley).

In 2001, Saramito and Roquet revisited the classical fully developed Poiseuille flow of a Bingham yield-stress fluid in pipe [12] with non-circular cross-section. Addressing the case of a square cross-section, they pointed out the lack of precision of the previous numerical computations, that were not able to compute accurately the yield surfaces that separate the shear region from the central plug and the dead zones. They proposed a new mixed anisotropic auto-adaptive finite element method coupled to the augmented Lagrangian algorithm. The mesh refinement is expected to capture accurately the free boundaries of the rigid zones. Based on *a priori* error estimate on adapted meshes, Roquet et al. [13] performed the numerical analysis of the method and showed that it converges with an optimal global order of accuracy. Finally, the extension of this approach to more general flows of a Bingham fluid is addressed in [14] where the authors considered the flow around a cylinder.

Slip occurs in the flow of two-phase systems, such as polymer solutions, emulsions, and particle suspensions, because of the displacement of the disperse phase away from solid bound-

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doi:10.1016/j.jnnfm.2007.12.003

aries [15]. It may be the result of either a static depletion effect at the solid boundaries or a shear induced particle migration. In any case, there is, close to the wall, a thin layer of fluid of lower viscosity than that of the bulk material, so that the shear amplitude is much larger in this layer than in the rest of the flow domain. This phenomenon appears to be more pronounced when the material possesses a yield stress, such as pastes [16–18]. It is known from experimental results [19] that wall slip occurs only when the wall shear stress exceeds a certain value. In practical viscoplastic flow problems such as concrete pumping (see e.g. [20,21]), it also appears that a no-slip boundary condition is not a satisfactory model. The fluid slips when the tangential stress exceeds a critical value, and, otherwise the fluid sticks at the wall. This critical value may be considered as an intrinsic characteristic of the material and its relation to the wall: in the following, it will be called the *yield-force* of the fluid.

The first attempt to formulate a slip yield boundary condition is due in 1965 to Pearson and Petrie [22]. In 1991, Fortin et al. [23], used it from a computational point of view for the flow of a Newtonian fluid for the sudden contraction geometry, and next, in 2004 by Roquet and Saramito [24] for the straight pipe flow with a square cross-section. In 1998, Huilgol [25] analyzed the variational principle of a yield-stress fluid together with a slip yield condition. From a mathematical point of view, there is an analogy of the slip of fluids on a wall and the slip solids on over surfaces. In the context of solid mechanics and contact problems, Coulomb type friction has been studied by many authors: refer e.g. to Haslinger et al. [26, p. 377] or Ionescu and Vernescu [27] for the numerical analysis and to Kikuchi and Oden [28] for the finite element approximation. In this case, the slip yield stress is no more a constant, and should be replaced by a quantity that depends upon the pressure at the boundary. Nevertheless, previous works do not study the stick–slip transition. In this paper, since our purpose is to study a new numerical algorithm for the stick–slip transition capturing, we suppose that the slip yield stress is a constant.

The aim of this paper is to extend the technique presented in Saramito and Roquet [12,24] in order to apply it to the flow of a Bingham fluid in a straight pipe with constant cross-section with the stick–slip law at the wall. In Section 2, all the governing laws of the flow model are presented, ending with the non-dimensional formulation of the flow of a Bingham fluid with the stick–slip law in a straight pipe. In the third section, the numerical method is described. The last section presents all the numerical results and the discussion. The role of the two dimensionless numbers associated to the yield parameters of the flow structure are investigated in detail. The computations cover the full range of the two main dimensionless numbers and exhibit complex flow patterns: all the different flow regimes are completely identified.

2. Problem statement

The general equations for the flow of a Bingham fluid with the stick–slip law is given first. Then, it is specialized for the case of a straight pipe with constant cross-section.

2.1. Constitutive equation and conservation laws

Let σ_{tot} denotes the total Cauchy stress tensor:

$$\sigma_{\text{tot}} = -p\, I + \sigma, \tag{1}$$

where σ denotes its deviatoric part, and p the pressure. In this paper, the fluid is supposed to be viscoplastic, and the relation between σ and $D(\mathbf{u})$ is given by the Bingham model [29,30]:

$$\begin{cases} \sigma = 2\eta D(\mathbf{u}) + \sigma_0 \dfrac{D(\mathbf{u})}{|D(\mathbf{u})|} & \text{when } D(\mathbf{u}) \neq 0, \\ |\sigma| \le \sigma_0 & \text{when } D(\mathbf{u}) = 0, \end{cases} \tag{2}$$

here $\sigma_0 \ge 0$ is the yield stress, $\eta > 0$ is the constant viscosity, $\mathbf{u}$ is the velocity field and $D(\mathbf{u}) = (\nabla\mathbf{u} + \nabla\mathbf{u}^{\text{T}})/2$. For any tensor $\tau = (\tau_{ij})$, the notation $|\tau|$ represents the matrix norm:

$$|\tau| = \left(\frac{\tau : \tau}{2}\right)^{1/2} = \frac{1}{\sqrt{2}}\left(\sum_{i,j}\tau_{ij}\right)^{1/2}. \tag{3}$$

The constitutive Eq. (2) writes equivalently:

$$D(\mathbf{u}) = \begin{cases} \left(1 - \dfrac{\sigma_0}{|\sigma|}\right)\dfrac{\sigma}{2\eta} & \text{when } |\sigma| > \sigma_0, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The slip boundary condition reads:

$$\mathbf{u_t} = \begin{cases} -\left(1 - \dfrac{s_0}{|\sigma_{\mathbf{nt}}|}\right)\dfrac{\sigma_{\mathbf{nt}}}{c_{\text{f}}} & \text{when } |\sigma_{\mathbf{nt}}| > s_0, \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

where $s_0 \ge 0$ the slip yield stress and $c_{\text{f}} > 0$ the friction dissipation coefficient. The notations $\mathbf{u_t}$ and $\sigma_{\mathbf{nt}}$ are defined by

$$\begin{aligned} \mathbf{u_t} &= \mathbf{u} - (\mathbf{u}.\mathbf{n})\,\mathbf{n}, \\ \sigma_{\mathbf{nt}} &= \sigma.\mathbf{n} - (\sigma_{\mathbf{nn}})\,\mathbf{n}, \end{aligned} \tag{6}$$

where $\sigma_{\mathbf{nn}} = (\sigma.\mathbf{n}).\mathbf{n}$ and $\mathbf{n}$ is the unit outward normal vector. For any vector field $\mathbf{v}$, the notation $|.|$ represents the vector norm $|\mathbf{v}| = (\mathbf{v}.\mathbf{v})^{1/2}$. Notice that the vector field $\sigma_{\mathbf{nt}}$ is tangent to the boundary and that $\sigma_{\mathbf{nn}}$ is a scalar field defined on the boundary. Observe the analogy of structure between the slip law (5) and the Bingham constitutive Eq. (4). The slip relation can be also written as:

$$\begin{cases} \sigma_{\mathbf{nt}} = -c_{\text{f}}\mathbf{u_t} - s_0\dfrac{\mathbf{u_t}}{|\mathbf{u_t}|} & \text{when } |\mathbf{u_t}| \neq 0, \\ |\sigma_{\mathbf{nt}}| \le s_0 & \text{when } |\mathbf{u_t}| = 0. \end{cases} \tag{7}$$

Again, observe the analogy between (7) and (2).

The boundary condition is complemented by a condition expressing that the fluid does not cross the boundary:

$$\mathbf{u} \cdot \mathbf{n} = 0. \tag{8}$$

We remark that for $s_0 = 0$, one obtains the classical linear slip boundary condition: the fluid slips for any non-vanishing shear stress $\sigma_{\mathbf{nt}}$. For $s_0 > 0$, boundary parts where the fluid sticks can be observed. As s_0 becomes larger, these stick regions develop.