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Re-entrant corner flows of UCM fluids: The Cartesian stress basis

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Abstract

For a given re-entrant corner geometry, we describe a two parameter family of solutions for the local asymptotic behaviour of the flow and stress fields of UCM fluids. The two parameters used are the coefficients of the upstream wall shear rate and pressure gradient. In describing this parametric solution, the relationship between the Cartesian and natural stress basis is explained, reconciling these two equivalent formulations for the problem. The asymptotic solution structure investigated here comprises an outer (core) region together with inner regions (single wall boundary layers) located at the upstream and downstream walls. It is implicitly assumed that there are no regions of recirculation at the upstream wall, i.e. we consider flow in the absence of a lip vortex. The essential feature of the analysis is a full description of the matching between the outer and inner regions in both the Cartesian and natural stress bases, as well as the derivation of numerical estimates of important solution parameters such as the coefficients of the stream function and extra-stresses in the outer (core flow) region together with the downstream wall shear rate. This work is divided into two papers, the first one describing the solution structure in the Cartesian stress basis for the core and upstream boundary layer, and the second paper using the natural stress basis which allows the downstream boundary layer solution to be linked through the core to the upstream boundary layer solution. It is the latter formulation of this problem which allows a complete solution description.

Keywords: UCM fluid; Re-entrant corner; Boundary layer; Self-similar solutions; Natural stress basis

1. Introduction

Substantial progress has been made recently on the local asymptotic analysis of the flow and stress fields associated with elastic fluids at re-entrant corners. The work of Rallinson and Hinch [7] used the natural stress basis formulation of Renardy [8,10], to analyze the boundary layer at the downstream wall for Oldroyd-B fluids. In contrast, Evans [1,3] used the Cartesian basis for the stress in the analysis of such a structure for UCM and Oldroyd-B fluids, respectively. This recent work completes the analysis of the asymptotic regions previously known to comprise of a core flow classifying the stress singularity for such fluids as given by Hinch [6] and the upstream wall boundary layer analysis of Renardy [9]. It is natural to seek to understand how these approaches are linked in order to obtain a comprehensive understanding of the solution structure. This paper and its companion [5] will be concerned with the UCM model, extending the analytical and numerical analysis in Rallinson and Hinch [7] and Evans [1]. Implicit to the assumed solution structure is the absence of a lip vortex at the upstream wall. Comments on the case when such a feature is present are given in the discussion section.

The goals then are to:

1. reconcile the approaches of using Cartesian and natural stress basis,
2. describe for the UCM model the local asymptotic solution for flow and stress fields around a re-entrant corner of given angle as a two parameter family of solutions. The two parameters chosen are the coefficients of the upstream wall shear rate (or shear stress) and pressure gradient.

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In addressing these goals, we will make extensive use of previous results. Many of the key mathematical results were determined in the seminal work of Hinch [6] and Renardy [8–10]. An important contribution of Rallinson and Hinch [7] was to show how these may be pieced together linking the solution between the different asymptotic regions to give a full description of the problem as well as elucidating the presence of an essential singularity in the boundary layer equations at the downstream wall.

The work has been divided into a pair of papers, with this one concentrating on the Cartesian formulation of the problem and a companion paper [5] the natural stress basis. The layout of this paper is as follows. In the rest of this introduction we record the main mathematical results upon which the local asymptotic theory is constructed. The outer solution is addressed in Section 2, whilst the boundary layer equations are analyzed in Section 3. The new results in these sections are mainly concerned with the far-field behaviour of the stream function and stress components in both the upstream and downstream boundary layers. Consequently, the full asymptotic expansion required in the core outer region can be determined as well as the corresponding behaviours for the natural stress variables, which extend the results in Evans [1]. Additional new results concern the downstream wall behaviour where an essential singularity is known to exist (see Rallinson and Hinch [7]), the precise behaviour of which is now obtained. The parametric solution description is then addressed in Section 4 where the upstream and downstream boundary layers are considered separately. This is the new aspect in the treatment of the re-entrant corner problem which completes the previous body of work. Numerical solution of the downstream layer is presented, but we leave the determination of key parameters such as the downstream wall shear rate to the natural stress basis formulation in the companion paper.

For planar steady flow, the governing equations in dimensionless form are the momentum equation

$$Re\,\mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + \nabla\cdot\mathbf{T} \tag{1.1}$$

the incompressibility condition

$$\nabla\cdot\mathbf{v} = 0 \tag{1.2}$$

and the constitutive relation

$$\mathbf{T} + We\,\overset{\triangledown}{\mathbf{T}} = 2\,\mathbf{D}, \tag{1.3}$$

with the upper convected (or Oldroyd) stress derivative defined by

$$\overset{\triangledown}{\mathbf{T}} = (\mathbf{v}\cdot\nabla)\mathbf{T} - (\nabla\mathbf{v})\mathbf{T} - \mathbf{T}(\nabla\mathbf{v})^{\mathrm{T}}$$

and

$$\mathbf{D} = \tfrac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}})$$

being the rate-of-strain tensor. Here, $\mathbf{T} = (T_{ij})$ is the extra-stress tensor, $\mathbf{v}$ is the velocity field, p is the pressure, Re is the Reynolds number and We is the Weissenberg number. The total stress tensor is given by $-p\mathbf{I} + \mathbf{T}$. These equations are taken to hold in the two dimensional sector $0 < r < \infty$, $0 \le \theta \le \pi/\alpha$ with $\theta = 0$ representing the upstream wall and $\theta = \pi/\alpha$ the downstream wall ($(1/2) \le \alpha < 1$). Here, r is the radial distance from the corner and θ the polar angle. Our intention is to study the local asymptotics of the Eqs. (1.1)–(1.3) as the corner is approached in the singular limit $r \to 0$. We now make the following preliminary remarks, before describing the detailed asymptotic structure of the solution.

Remark 1 *(Cartesian formulation).* Initially, Cartesian axes are taken with the x-axis along the upstream wall $\theta = 0$, and the y-axis along the ray $\theta = \pi/2$. Further, introducing the stream function ψ, the Eqs. (1.1) and (1.3) can be written in the form

$$Re\left(\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}\right) = -\frac{\partial p}{\partial x} + \frac{\partial T_{11}}{\partial x} + \frac{\partial T_{12}}{\partial y}, \tag{1.4}$$

$$Re\left(-\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x^2} + \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y\partial x}\right) = -\frac{\partial p}{\partial y} + \frac{\partial T_{12}}{\partial x} + \frac{\partial T_{22}}{\partial y}, \tag{1.5}$$

and

$$T_{11} + We\left(\frac{\partial\psi}{\partial y}\frac{\partial T_{11}}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T_{11}}{\partial y} - 2\frac{\partial^2\psi}{\partial y^2}T_{12} - 2\frac{\partial^2\psi}{\partial x\partial y}T_{11}\right) = 2\frac{\partial^2\psi}{\partial x\partial y}, \tag{1.6}$$

$$T_{22} + We\left(\frac{\partial\psi}{\partial i}\frac{\partial T_{22}}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T_{22}}{\partial y} + 2\frac{\partial^2\psi}{\partial x^2}T_{12} + 2\frac{\partial^2\psi}{\partial x\partial y}T_{22}\right) = -2\frac{\partial^2\psi}{\partial x\partial y}, \tag{1.7}$$

$$T_{12} + We\left(\frac{\partial\psi}{\partial y}\frac{\partial T_{12}}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T_{12}}{\partial y} + \frac{\partial^2\psi}{\partial x^2}T_{11} - \frac{\partial^2\psi}{\partial y^2}T_{22}\right) = \frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}. \tag{1.8}$$