ELSEVIER
J. Non-Newtonian Fluid Mech. 136 (2006) 106–117
Journal of Non-Newtonian Fluid Mechanics
www.elsevier.com/locate/jnnfm

On the importance of the pressure dependence of viscosity in steady non-isothermal shearing flows of compressible and incompressible fluids and in the isothermal fountain flow ☆

R.R. Huilgol*, Z. You

School of Informatics and Engineering, Flinders University of South Australia, GPO Box 2100, Adelaide, SA 5001, Australia

Received 30 September 2005; received in revised form 7 March 2006; accepted 7 March 2006

Abstract

Four steady non-isothermal viscometric flows of a class of compressible and incompressible fluids with a viscosity depending on the shear rate, pressure and temperature are considered and consistency relations between the pressure gradient in the flow direction and that in the direction of the velocity gradient are derived. In particular, the dependence of the viscosity on the pressure plays the most significant role in these relations. Examples of pressure fields obeying these compatibility conditions are given and their significance is examined. In particular, it is shown that unidirectional flows in compressible and incompressible fluids are the exception rather than the rule when the viscosity depends on pressure. Motivated by this, we examine the fountain flow effect for an incompressible fluid with a pressure dependent viscosity under isothermal conditions using the finite element method.

Keywords: Compressible fluids; Viscosity; Pressure; Temperature; Shear rate; Compatibility conditions; Fountain flow

1. Introduction

Consider four steady velocity fields: channel flow, simple shearing flow, Poiseuille flow and Couette flow. It is well known that these are possible in every incompressible fluid if the viscosity η depends on the shear rate $\dot{\gamma} \geq 0$. Suppose that the viscosity depends on the absolute temperature T as well, i.e., let $\eta = \eta(T, \dot{\gamma})$. If we assume that in each one of the flows, the temperature gradient is parallel to the velocity gradient, one can show that the assumed velocity and temperature fields are consistent with the equations of motion and the balance of energy equation. For example, see the book by Bird et al. [1] on transport phenomena where such problems are discussed.

Now, let us suppose that the viscosity depends on a third variable, viz., the pressure p. That is, $\eta = \eta(p, T, \dot{\gamma})$. Do the assumed velocity and temperature fields satisfy the equations of motion? It will be shown below that in each flow, new features arise. For example, in the channel flow discussed in Section 3, it is found that the pressure gradient in the flow direction induces a second one in the direction of the velocity gradient; from this, it follows that normal stress effects appear. If the viscosity η depends on the pressure p only and is of the form $\eta = \alpha p$, where $\alpha > 0$ is a constant, no flow can occur due to a constant pressure gradient in the flow direction; this is also true for the Poiseuille flow discussed in Section 5. However, under an exponentially diminishing gradient, a channel flow can exist which turns out to be physically unacceptable. if $\eta = f(T) + (p/\dot{\gamma})$, then a rectilinear flow cannot occur in a channel or in a pipe of circular cross-section (see Sections 3 and 5). However, a simple shearing flow is feasible provided the temperature field satisfies a simple condition (see Section 4).

Turning to the Couette flow in Section 6, it is shown there that the velocity field $\dot{r} = 0$, $\dot{\theta} = \omega(r)$, $\dot{z} = 0$, and the temperature field $T = T(r)$ are consistent with the equations of motion because the pressure gradient $\partial p/\partial\theta = 0$. This result that a pressure gradient in the radial direction does not induce one in the azimuthal direction is in contrast with the channel and Poiseuille flows studied here. It arises solely out of the presence of the inertial term in the radial direction. Thus, it appears that rectilinear flows in fluids with a pressure dependent viscosity

☆ Some parts of the work reported here were presented at the Banff International Research Station for Mathematical Innovation and Discovery (BIRS) workshop on "Visco-plastic fluids: from Theory to Application", held from October 22–27, 2005.

* Corresponding author. Tel.: +61 8 8201 2938; fax: +61 8 8201 2904.
E-mail address: Raj.Huilgol@flinders.edu.au (R.R. Huilgol).

0377-0257/$ – see front matter
doi:10.1016/j.jnnfm.2006.03.011

is less likely than two-dimensional flows. That is, secondary flows are likely to occur.

From the foregoing, it is obvious that temperature and shear rate dependence of the viscosity are not problematic; only the pressure dependence is. Hence, one of the aims of this work is to explore the connection between the two mutually orthogonal pressure gradients when they exist and the impact they have on a given flow in question due to the pressure dependence of viscosity. Before investigating the importance of this property, one has to ask a fundamental question: can the viscosity in an incompressible fluid be a function of pressure? The reason is that in the traditional approach to continuum mechanics, the pressure is defined as an entity which produces zero stress power in isochoric motions (for example, see Section 33 of Huilgol and Phan-Thien [2]). Hence, it is pleasing to discover through the researches of Antman [3], and Antman and Marlow [4] that material functions, such as viscosity and normal stress differences, can depend on the indeterminate part of the stress tensor, viz., the pressure term. Hence, our study of incompressible fluids with a pressure dependent viscosity is compatible with their conclusions. See Hron et al. [5] as well for a thorough discussion of this matter.

Now, let us enlarge the context further by assuming that the fluid is compressible. Do the conclusions regarding the four shearing flows hold true here? In order to answer this, we consider a class of compressible fluids whose material properties depend on pressure, temperature and the three invariants of the first Rivlin-Ericksen tensor [6]. By assuming that the component of the stress tensor, which is isotropic and different from the pressure term, vanishes in shearing flows, we will show that the equations of motion are identical for both compressible and incompressible fluids. Thus, conclusions reached about the former apply to the latter with minor changes. Extensions of some of the results to the Criminale–Ericksen–Filbey model [7] and viscoplastic fluids are made in Sections 7 and 8.

Finally, having demonstrated that the pressure dependence of the viscosity is highly significant, we examine its role in the fountain flow effect in Section 9 using finite elements, and show that it is equivalent to the effect found in shear thinning fluids for the two models studied here.

2. The constitutive relations

In order to examine compressibility effects, one approach is to assume that the stress tensor depends on ρ, which is the density, the absolute temperature $T > 0$, along with some kinematical variables. For example, the Stokesian fluids described by Eringen ([8], Section 5.7) are defined in the above manner. That is, one defines the total stress tensor $\mathbf{T}$ through:

$$\mathbf{T} = -p\mathbf{1} + \mathbf{S}. \tag{2.1}$$

Here, the 'thermodynamic pressure' p, the density ρ and the temperature T satisfy an equation of state:

$$f(p, \rho, T) = 0. \tag{2.2}$$

From the above implicit equation, if one assumes that one can solve for the pressure term as $p = p(\rho, T)$, we have five quantities, viz., ρ, T, and the three components of the velocity field $\mathbf{v}$, to be found from the five equations of continuum mechanics. These are the continuity equation, the three equations of motion and the balance of energy equation. Of course, in order to solve these, it is essential to specify the constitutive relations for the extra stress tensor $\mathbf{S}$, the internal energy and the heat flux vector [1,2,8]. Naturally, $\mathbf{S}$ will depend on ρ, T and some kinematical variables.

In the methodology adopted here, we suppose that we can solve Eq. (2.2) for the density, i.e., $\rho = \rho(p, T)$. Then, the five unknowns become p, T, $\mathbf{v}$. In turn, this suggests that we define $\mathbf{S}$ in terms of p, T, and the relevant kinematical variables. In analogy with the Stokesian fluids, we shall study a class of compressible fluids of the form:

$$\mathbf{S} = \hat{\lambda}(p, T, I_1, I_2, I_3)\mathbf{1} + \hat{\eta}(p, T, I_1, I_2, I_3)\mathbf{A}_1, \tag{2.3}$$

where $\mathbf{A}_1$ is the first Rivlin–Ericksen tensor [6]. That is, in terms of the velocity field $\mathbf{v}$ and its gradient $\mathbf{L} = \nabla\mathbf{v}$, we define:

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^{\mathrm{T}}, \tag{2.4}$$

with $\mathbf{L}^{\mathrm{T}}$ being the transpose of $\mathbf{L}$. Moreover, in Eq. (2.3), I_1, I_2, I_3 are the following three invariants of $\mathbf{A}_1$:

$$I_1 = \mathrm{tr}\,\mathbf{A}_1, \qquad I_2 = \mathrm{tr}\,\mathbf{A}_1^2, \qquad I_3 = \mathrm{tr}\,\mathbf{A}_1^3, \tag{2.5}$$

with tr being the trace operator.

Now, in shearing flows, it is well known that both $I_1 = 0$, and $I_3 = 0$. Thus, if we wish to obtain equations of motion which are identical for compressible and incompressible fluids in this class of flows, it is necessary to demand that

$$\hat{\lambda}(p, T, 0, I_2, 0) = 0. \tag{2.6}$$

Next, we shall assume that in shearing flows:

$$\hat{\eta}(p, T, 0, I_2, 0) = \eta(p, T, \sqrt{I_2/2}) = \eta(p, T, \dot{\gamma}), \tag{2.7}$$

where $\dot{\gamma} \geq 0$ is the shear rate.

In sum, the compressible fluid has the following constitutive relation in shearing flows:

$$\mathbf{T} = -p\mathbf{1} + \eta(p, T, \dot{\gamma})\mathbf{A}_1. \tag{2.8}$$

The incompressible fluid has the same relation except that p does not satisfy a separate equation of state.

In addition, the thermodynamic pressure in a compressible fluid must always be positive, i.e., $p > 0$. While this is not required in an incompressible fluid since the datum can be set to zero anywhere in a flow, it is a requirement that $\eta > 0$ when we prescribe the viscosity η as a function of p. These observations play a crucial role in the sequel, although the impact is studied extensively for the case of the channel flow only.

3. Channel flow

We shall assume that a steady channel flow occurs in the x-direction with a velocity field $u = u(y)$, where $-1 \leq y \leq 1$. The boundary conditions are such that $u(\pm 1) = 0$. Consequently, we assume that $\mathrm{d}u/\mathrm{d}y = u'(y) > 0$ in $-1 < y < 0$, so that $\dot{\gamma} = u'(y)$; in $0 < y < 1$, clearly $\dot{\gamma} = -\mathrm{d}u/\mathrm{d}y$, of course. In addition, we let the temperature field $T = T(y)$ as well.