Engineering Structures 173 (2018) 383–392

Contents lists available at ScienceDirect

Engineering Structures

journal homepage: www.elsevier.com/locate/engstruct

Nonlinear structural dynamic analysis by a stabilized central difference method

Delfim Soares Jr.[a,*], Georg Großeholz[b]

[a] Structural Engineering Department, Federal University of Juiz de Fora, Juiz de Fora, MG, Brazil
[b] Institute of Modelling and Computation, Hamburg University of Technology, Hamburg, Germany

ARTICLE INFO

Keywords:
Time integration
Stabilized central difference method
Nonlinear structural dynamics
Accuracy
Stability

ABSTRACT

In this work, a simple stabilized central difference technique is discussed, to analyse nonlinear models. The proposed technique is unconditionally stable for linear systems and it provides enhanced stability features for nonlinear applications. The proposed method stands as a direct single step procedure, avoiding any iterative computations when solving nonlinear models; thus, it is very efficient. In addition, it is extremely accurate, providing much reduced period elongation and amplitude decay errors, compared to standard methods. The present work also introduces a criterion for updating the nonlinear system matrices, significantly reducing the computational complexity of the simulation and enhancing the efficiency of the technique. Numerical results are presented along the manuscript, illustrating the performance and accuracy of the proposed methodology.

1. Introduction

The method of lines [1] is a commonly used technique to simulate structural dynamic phenomena numerically. Here, the governing equations are first semi-discretized in the spatial domain and the resulting system of ordinary differential equations (ODE) is solved by an appropriate time-marching scheme, afterwards. Inherently, the mathematical complexity of the described simulation process grows disproportionately faster compared to the macroscopic complexity of the problem in focus and, consequently, a reasonable balance between the accuracy and the numerical costs of the simulation has to be kept. The propagation of the approximate solution in time represents a big part of the computational cost necessary for the simulation process, especially when non-linear behaviour is taken into account. Therefore, efficient time-marching procedures are crucial in order to obtain reliable results at a reasonable cost. In non-linear structural dynamics, usually direct integration methods are applied. These can be subdivided in two families, namely explicit and implicit procedures. In an implicit approach, the constitutive variables are expressed as functions of the current time of analysis. In this case, when nonlinear models are focused, the solution has to be carried out by iterative schemes, such as the Newton-Raphson algorithm [2], which always cause a significant computational overhead. Furthermore, these schemes may also have a negative impact on the convergence behavior and consequently the application of an implicit method can be computationally very demanding. In explicit approaches, on the other hand, all constitutive variables are available from computations at previous time-steps. Hence, the calculations necessary to propagate the approximate solution one increment further in time are quite simple and – in case lumped mass and damping matrices are considered – usually do not require the solution of any linear system of equations. However, due to conditional stability, classic explicit methods often imply restrictive limitations for the incremental time step size Δt, resulting once again in high computational costs.

Classic explicit time stepping schemes are often modified by manipulating the system matrices, which result from the spatial semi-discretization, in order to improve stability properties. For example, in classic mass scaling approaches [3–5], an explicit time marching procedure is applied in combination with a perturbed system mass matrix. This influences the inertia and consequently the natural frequency of the discretized system, and may result in a larger critical time increment [6]. These methods are easy to add to an existing explicit simulation environment and they provide good results in many cases. However, if the time increment is not taken into account for the choice of the scaling matrix, convergence is not guaranteed anymore and unreliable results might be obtained. Another approach to enhance stability is the design of so-called hybrid methods, such as the Rosenbrock-Wanner methods, described in [7,8], the family of procedures presented by Tamma et al. [9–11], or the stabilized central difference scheme, discussed in [12]. These methods are often referred to as ‘nonlinearly

* Corresponding author.
E-mail address: delfim.soares@ufjf.edu.br (D. Soares).

https://doi.org/10.1016/j.engstruct.2018.06.115
Received 12 December 2017; Received in revised form 20 June 2018; Accepted 28 June 2018

explicit', since they may require the solution of a linear system in the linear case, but do not involve any nonlinear iterative process if nonlinear phenomena are considered. In these cases, an approximation for the stiffness of the system is still required at each time step. Thus, computationally expensive updating procedures are necessary. Motivated by the method presented in [12], the present work introduces an update criterion for the approximate stiffness, significantly reducing the computational complexity for the simulation process. By the application of an approximate stability condition, taking into account the softening and hardening behaviour of the considered model, a simple, adaptive update criterion is established, which renders an efficient and very competitive time marching technique. In addition, the methodology discussed in [12] is further generalized in this work, allowing considering physical damping in the model, as well as exploring more complex nonlinear systems.

In the next section, the stabilized central difference scheme is briefly summarized, followed by a stability analysis, the deduction of the new update criterion, and the resulting algorithm. In Section 3, different linear and nonlinear numerical examples are presented, illustrating the performance and potentialities of the proposed technique.

2. The stabilized central difference method

The governing system of equations describing the non-linear dynamic model in focus is given by [13]:

$$\mathbf{M}\ddot{\mathbf{U}}(t) + \mathbf{C}\dot{\mathbf{U}}(t) + \mathbf{P}(t) = \mathbf{F}(t) \tag{1}$$

where $\mathbf{M}$ and $\mathbf{C}$ stand for the mass and damping matrices, respectively, resulting from the spatial discretization, $\mathbf{P}(t)$ denotes the vector of nodal point forces corresponding to the element internal stresses, and $\mathbf{F}(t)$ stands for the vector of externally applied nodal point forces. $\mathbf{P}(t)$ is a function of $\mathbf{U}(t)$. and $\mathbf{U}(t)$, $\dot{\mathbf{U}}(t)$ and $\ddot{\mathbf{U}}(t)$ represent displacement, velocity and acceleration vectors, respectively. In linear analysis, $\mathbf{P}(t)$ is usually represented as $\mathbf{KU}(t)$, where $\mathbf{K}$ stands for the linear stiffness matrix of the model.

In the proposed stabilized approach, the mass matrix of the model is modified, incorporating stability to the classical central difference (CD) methodology. Thus, a new mass matrix $\mathbf{M}'$ is considered, which is formulated as follows:

$$\mathbf{M}' = \mathbf{M} + \frac{1}{2}\Delta t\mathbf{C} + a\Delta t^2\mathbf{K}_T \tag{2}$$

where $\mathbf{K}_T$ stands for the tangent nonlinear stiffness matrix, Δt is the adopted time-step and a stands for a time integration parameter. Eq. (2) stands as an extended development of the modified mass matrix presented in [12]. In this new modified mass matrix, an extra term, which considers the damping matrix of the model, is proposed, enhancing the generality and the benefits of the stabilized method.

Taking into account the standard central difference method,

$$\ddot{\mathbf{U}}^n = \frac{1}{\Delta t^2}(\mathbf{U}^{n+1} - 2\mathbf{U}^n + \mathbf{U}^{n-1}) \tag{3a}$$

$$\dot{\mathbf{U}}^n = \frac{1}{2\Delta t}(\mathbf{U}^{n+1} - \mathbf{U}^{n-1}) \tag{3b}$$

and the new mass matrix (Eq. (2)), the time domain solution of Eq. (1) can be expressed as indicated in Eq. (4), once Eqs. (2) and (3) are applied to Eq. (1) at time step n:

$$\mathbf{U}^{n+1} = 2\mathbf{U}^n - \mathbf{U}^{n-1} + (\mathbf{M} + \Delta t\mathbf{C} + a\Delta t^2\mathbf{K}_T)^{-1}[\Delta t^2(\mathbf{F}^n - \mathbf{P}^n) - \Delta t\mathbf{C}(\mathbf{U}^n - \mathbf{U}^{n-1})] \tag{4}$$

Eq. (4) defines the proposed new time marching technique. As one can observe, the proposed technique stands as a simple single-step procedure, allowing to solve the non-linear model directly; i.e., without any iterative process or any sub-step routines. Thus, the proposed approach exhibits a good level of efficiency.

In Eq. (4), the tangent stiffness matrix $\mathbf{K}_T$ is related to the time instant t^n (as well as the internal force $\mathbf{P}$), and consequently it has to be re-evaluated at each time step of the nonlinear analysis for a consistent formulation. However, in order to avoid this computationally demanding procedure, an updating criterion is discussed in the next subsections, allowing to recalculate $\mathbf{K}_T$ just when necessary to ensure stability. Thus, $\mathbf{K}_T$ may be computed at fewer time steps (or even just once), enhancing the efficiency of the technique. An expression for the time integration parameter a is also discussed in the next subsection, enabling a calculation according to the properties of the model, rendering a more accurate technique.

2.1. Stability analysis

In order to gain insight into the stability properties of the proposed technique, linear analysis for a single degree of freedom (SDOF) model is considered here [14,15]. This SDOF model is associated to the modal analysis of the problem, taking into account its critical mode. Thus, in this analogy, $K_T \equiv \omega^2$, where ω represents the natural frequency of the mode, $C \equiv 2\xi\omega$, where ξ represents the damping ratio, and $M \equiv 1$. In this context, if the nonlinear internal force term at the time instant t^n is represented as K'U, the amplification matrix of the method can be defined following standard approaches [15], and it can be stated as indicated below, taking into account the proposed stabilized central difference method:

$$\mathbf{A} = \begin{bmatrix} \frac{2M + \Delta tC + 2a\Delta t^2K_T - \Delta t^2K'}{M + \Delta tC + a\Delta t^2K_T} & -\frac{M + a\Delta t^2K_T}{M + \Delta tC + a\Delta t^2K_T} \\ 1 & 0 \end{bmatrix} \tag{5}$$

where its eigenvalues may be expressed as:

$$\lambda = \frac{2M + \Delta tC + 2a\Delta t^2K_T - \Delta t^2K' \pm \Delta t\sqrt{C^2 - 4MK' - 2\Delta tCK' - 4a\Delta t^2K_TK' + \Delta t^2K'^2}}{2M + 2\Delta tC + 2a\Delta t^2K_T} \tag{6}$$

If complex eigenvalues take place, i.e., if relation (7) is true,

$$(C^2 - 4MK' - 2\Delta tCK' - 4a\Delta t^2K_TK' + \Delta t^2K'^2) < 0 \tag{7}$$

the spectral radius of the method can be established as:

$$\rho^2 = \frac{(2M + \Delta tC + 2a\Delta t^2K_T - \Delta t^2K')^2 - \Delta t^2(C^2 - 4MK' - 2\Delta tCK' - 4a\Delta t^2K_TK' + \Delta t^2K'^2)}{(2M + 2\Delta tC + 2a\Delta t^2K_T)^2} = \frac{M + a\Delta t^2K_T}{M + a\Delta t^2K_T + \Delta tC} \tag{8}$$

which is less or equal 1, ensuring stability in the linear case (in fact, $\rho < 1$ for $C \neq 0$, and $\rho = 1$ for $C = 0$; featuring a non-dissipative time marching numerical approach). Thus, if relation (7) is followed, stable results can be expected.

Taking into account an undamped model (i.e., $C = 0$), which stands for the most critical configuration for stability, and considering updated values for K_T (i.e., $K_T \equiv K'$), an expression for a can be established to ensure relation (7). Of course, in this context, relation (7) is easily followed by adopting $a = \frac{1}{4}$. However, in order to obtain a more accurate numerical technique, it is interesting to select lower values for a. In this case, reduced period elongation errors will occur, and an enhanced technique will arise. In this work, the following expression is adopted for a:

$$a = \frac{1}{4}\tanh\left(\frac{1}{4}\omega\Delta t\right) \tag{9}$$

which provides $a \in \left(0, \frac{1}{4}\right)$. In Eq. (9), ω stands for the maximal natural frequency of the model (or an upper approximation of it), which is computed from $\mathbf{M}$ and $\mathbf{K}_T$. Thus, the time integration parameter adapts itself according to the evolution of the properties of the model.

In Fig. 1, $\gamma = \xi(\xi - \Omega) - 1 - (a - 1/4)\Omega^2$, which represents relation (7)