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Enhanced moment and shear recovery for classic beams and frames

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ABSTRACT

From the early days of finite element analysis the numerical static condensation method has been applied to elements with internal nodes to reduce their memory requirements and to simplify their assembly. The static condensation method was first used by Wilson (1965) to eliminate the internal degrees of freedom in a quadrilateral finite element constructed from four triangles. This study applies modern symbolic analysis software to static condensation to reveal additional observations about its benefits and efficiency. It is shown that the stiffness matrix and force vector transformation matrices can be written in closed form for a given element type and used for more efficient post-processing. In addition, it is shown that the classic cubic beam and linear bar element matrices are the symbolic static condensation versions of the three node quintic beam and quadratic bar, respectively. That allows the cubic beam element to be easily post-processed to give cubic moment and quadratic moment and shear diagrams, respectively.

1. Introduction

The classic cubic beam and frame finite element is used in most of the software for beam and frame analysis despite the fact that it gives very poor estimates of the distribution of the moment and shear, which are important design factors. The practical approach to overcoming that flaw has been to divide each frame or beam span into forty of fifty cubic elements. In most common support and load cases a single quintic beam element will give the exact thin beam theory solution for deflection, slope, moment, and shear over the length of the same span. Yet, the quintic beam element is not commonly used. That may be simply due to the cubic beam being programmed first and/or the inconvenience of inputting three nodes instead of two.

It will be shown below that the two node classic beam element is the symbolic static condensation of the three node quintic beam element. After the system displacements of a cubic beam structure have been found the symbolic static condensation transformation matrices can be used expanding its four degrees of freedom to the six degrees of freedom of the quintic beam element. That enhances the accuracy of post-processing each beam element by allowing plots of fifth-degree displacements, cubic moment diagrams and quadratic shear force diagrams. That also means that spans between discontinuities in loads or stiffness can be modelled by a single classic beam instead of dozens.

2. Material and methods

The cubic and quintic for formulating beam and frame elements have appeared in the literature several times, for example: [1,2,3]. The elastic stiffness, geometric stiffness, elastic foundation stiffness, line load resultant, thermal (through depth) load matrices give an element matrix equilibrium equation of the cubic element as:

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} + \frac{N}{30L}\begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}\cdots$$
$$+\frac{kL}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ 13L & -3L^2 & 22L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}\cdots$$
$$=\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} + \frac{L}{60}\begin{bmatrix} 21 & 9 \\ 3L & 2L \\ 9 & 21 \\ -2L & -3L \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} + \frac{\alpha \Delta TEI}{t}\begin{Bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{Bmatrix} \tag{1}$$

where L = beam length, E = elastic modulus, I = moment of inertia, N = the axial load (positive in tension), k = the Winkler elastic foun-

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dation stiffness, f_k = line load value at node k, α = the coefficient of thermal expansion, ΔT = the linear increase in temperature from the bottom to the top of the beam, t = the depth of the beam, V_k = the external transverse shear force at node k, M_k = the external bending couple at node k, and the generalized displacements of the cubic element's degrees of freedom are $\delta^{e^T} = [v_1 \ \ \theta_1 \ \ v_2 \ \ \theta_2]$.

The corresponding matrix relation for the three-node quintic, six degree of freedom, beam element have been given in a number of references, including [2,3]:

$$\frac{EI}{35L^3}\begin{bmatrix} 5,092 & 1,138L & -3,584 & 1,920L & -1,508 & 242L \\ 1,138L & 332L^2 & -896L & 320L^2 & -242L & 38L^2 \\ -3,584 & -896L & 7,168 & 0 & -3,584 & 896L \\ 1,920L & 320L^2 & 0 & 1,280L^2 & -1,920L & 320L^2 \\ -1,508 & -242L & -3,584 & 1,920L & 5,092 & -1,138L \\ 242L & 38L^2 & 896L & 320L^2 & -1,138L & 332L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix}\cdots$$
$$+\frac{N}{630L}\begin{bmatrix} 1,668 & 39L & -1,536 & 240L & -132 & 9L \\ 39L & 28L^2 & -48L & 8L^2 & 9L & 5L^2 \\ -1,536 & -48L & 3,072 & 0 & -1,536 & 48L \\ 240L & 8L^2 & 0 & 256L^2 & -240L & -8L^2 \\ -132 & 9L & -1,536 & -240L & 1,668 & -39L \\ 9L & 5L^2 & 48L & -8L^2 & -39L & 28L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix}\cdots$$
$$+\frac{kL}{13,860}\begin{bmatrix} 2,092 & 114L & 880 & -160L & 262 & -29L \\ 114L & 8L^2 & 88L & -12L^2 & 29L & -3L^2 \\ 880 & 88L & 5,632 & 0 & 880 & 88L \\ -160L & -12L^2 & 0 & 128L^2 & 160L & -12L^2 \\ 262 & 29L & 880 & -160L & 2,092 & -114L \\ -29L & -3L^2 & -88L & -12L^2 & -114L & 8L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix}\cdots$$
$$=\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \\ V_3 \\ M_3 \end{Bmatrix}+\frac{L}{420}\begin{bmatrix} 57 & 44 & -3 \\ 3L & 4L & 0 \\ 16 & 192 & 16 \\ -8L & 0 & 8L \\ -3 & 44 & 57 \\ 0 & -4L & -3L \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}+\frac{\alpha\Delta TEI}{t}\begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{Bmatrix} \tag{2}$$

3. Theory

The static condensation algorithm of Professor Ed Wilson [4,5] is well known. The above Wilson static matrix equilibrium equation for an element: $[\boldsymbol{K_E} + \boldsymbol{K_N} + \boldsymbol{K_k}]\delta = \{\boldsymbol{F_P} + \boldsymbol{F_f} + \boldsymbol{F_T}\}$, or $\boldsymbol{K\delta} = \boldsymbol{F}$ can be written in a partitioned form to separate internal and external degrees of freedom. The first seven equations of reference [5] are re-written for this study as:

$$\begin{bmatrix} \boldsymbol{K_{aa}} & \boldsymbol{K_{ab}} \\ \boldsymbol{K_{ba}} & \boldsymbol{K_{bb}} \end{bmatrix}\begin{Bmatrix} \boldsymbol{\delta_a} \\ \boldsymbol{\delta_b} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{F_a} \\ \boldsymbol{F_b} \end{Bmatrix} \tag{3}$$

where $\boldsymbol{\delta_a}$ indicates the internal degrees of freedom to be eliminated and $\boldsymbol{\delta_b}$ indicates the external (end) degrees of freedom that are associated with the reduced stiffness matrix ($\boldsymbol{K}^*$). The upper partition gives

$$\boldsymbol{\delta_a} = \boldsymbol{K_{aa}^{-1}}\{\boldsymbol{F_a} - \boldsymbol{K_{ab}\delta_b}\}.$$

So the lower partition becomes

$$\boldsymbol{K_{bb}\delta_b} = \boldsymbol{F_b} - \boldsymbol{K_{ba}\delta_a} = \boldsymbol{F_b} - \boldsymbol{K_{ba}K_{aa}^{-1}}\{\boldsymbol{F_a} - \boldsymbol{K_{ab}\delta_b}\}$$

or

$$[\boldsymbol{K_{bb}} - \boldsymbol{K_{ba}K_{aa}^{-1}K_{ab}}]\boldsymbol{\delta_b} = \{\boldsymbol{F_b} - \boldsymbol{K_{ba}K_{aa}^{-1}F_a}\}$$

this defines the reduced element equilibrium equation as

$$\boldsymbol{K^*\delta_b} = \boldsymbol{F}^*. \tag{4}$$

Changing the Wilson notation slightly, this process defines a square work matrix, a rectangular transformation matrix, and a column transformation matrix of

$$\boldsymbol{W} \equiv \boldsymbol{K_{aa}^{-1}}, \ \ \boldsymbol{T_k} \equiv \boldsymbol{WK_{ab}}, \ \ \boldsymbol{T_F} \equiv \boldsymbol{WF_a}, \tag{5}$$

respectively, to denote the reduced system matrices as

$$\boldsymbol{K}^* = \boldsymbol{K_{bb}} - \boldsymbol{K_{ba}T_k}, \ \ \boldsymbol{F}^* = \boldsymbol{F_b} - \boldsymbol{K_{ba}T_F}. \tag{6}$$

Also, the first partition is expressed in terms of the transformations as

$$\boldsymbol{\delta_a} = \boldsymbol{T_F} - \boldsymbol{T_k\delta_b}. \tag{7}$$

Eqs. (5) and (6) are well suited for both symbolic and numerical calculation of the reduced element matrices. Wilson also gives an equation for recovering element stresses but notes "… in the case where a large number of stresses are required, it may be more efficient … to re-calculate the internal displacements" $\boldsymbol{\delta_a}$ "from the equivalent of equation", which is Eq. (7). That approach is utilized here.

A reformatted numerical static condensation algorithm (see the Matlab script in Appendix A) returns (or preferably stores) the two transformation matrices as well as the reduced element matrices. After the solution of the assembled system equations the element displacements, $\boldsymbol{\delta_b}$, are gathered and the eliminated internal degrees of freedom, $\boldsymbol{\delta_a}$, are recovered from Eq. (7) using the two element transformation matrices. An algorithm to numerically recover the internal degrees of freedom is shown in Appendix B.

Today, it is practical to execute the static condensation, for any element, using symbolic software to complete the algebra operations. Applying the above static condensation symbolically to the analytic expression for the quintic elastic stiffness matrix surprisingly gives the exact analytic expression for the well-known cubic beam element elastic stiffness in Eq. (1). Applying the above static condensation symbolically to the analytic expression for the quintic element resultant trapezoidal line load (with $w_2 = (w_1 + w_3)/2$) in Eq. (2) gives the exact analytic expression for the cubic beam element resultant trapezoidal line load in Eq. (1). Likewise, applying the above static condensation symbolically to the analytic expression for the quintic element thermal load resultant in Eq. (2) gives the exact analytic expression for the cubic beam element thermal load resultant in Eq. (1). Tong's Theorem [6] shows that the cubic element will give analytically exact nodal solutions when there is no axial load and no elastic foundation. Thus, the above symbolic condensations imply that the recovery of the eliminated internal node degrees of freedom may also prove to be exact, but they at least expected to be accurate. The generated condensation matrices for the three-node beam were

$$\boldsymbol{W} = \frac{L}{1024EI}\begin{bmatrix} 5L^2 & 0 \\ 0 & 28 \end{bmatrix}, \ \ \boldsymbol{T_k} = \frac{1}{8L}\begin{bmatrix} -4L & -L^2 & -4L & L^2 \\ 12 & 2L & -12 & 2L \end{bmatrix},$$
$$\boldsymbol{T_f} = \frac{L^3}{3840EI}\begin{Bmatrix} 5L(w_2 + w_1) \\ 2(w_2 - w_1) \end{Bmatrix} \tag{8}$$

The first to matrices are the same for any quintic beam, but the last one is only valid for a trapezoid line load. For other load cases it is simply $\boldsymbol{T_f} = \boldsymbol{WF_a}$ where the element load vector segment $\boldsymbol{F_a}$ was obtained by numerical integration or symbolic integration.