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Vibration analysis of rotating toroidal shell by the Rayleigh-Ritz method and Fourier series

Ivo Senjanović*, Neven Alujević, Ivan Ćatipović, Damjan Čakmak, Nikola Vladimir

University of Zagreb, Faculty of Mechanical Engineering and Naval Architecture, Zagreb, Croatia

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ABSTRACT

In this self-contained paper free vibrations of a pressurised toroidal shell, rotating around its axis of symmetry, are considered. Extensional and bending strain-displacement relationships are derived from general expressions for a thin shell of revolution. The strain and kinetic energies are determined in the co-rotating reference frame. The strain energy is first specified for large deformations and then split into a linear and a non-linear part. The nonlinear part, which is afterwards linearized, is necessary in order to take into account the effects of centrifugal and pressure pre-tensions. Both the Green-Lagrange nonlinear strains and the engineering strains are considered. The kinetic energy is formulated taking into account centrifugal and Coriolis terms. The variation of displacements u, v and w in the circumferential direction is described exactly. This is done by assuming appropriate trigonometric functions with a unique argument $n\varphi + \omega t$ in order to allow for rotating mode shapes. The dependence of the displacements on the meridional coordinate is described through Fourier series. The Rayleigh-Ritz method is applied to determine the Fourier coefficients. As a result, an ordinary stiffness matrix, a geometric stiffness matrix due to pressurisation and centrifugal forces, and three inertia matrices incorporating squares of natural frequencies, products of rotational speed and natural frequencies and squares of the rotational speed are derived. The application of the developed procedure is illustrated in the cases of a closed toroidal shell and a thin-walled toroidal ring. With the increase of the rotation speed the natural frequencies of most natural modes are split into two (bifurcate). The corresponding stationary modes are split into two modes rotating forwards and backwards around the circumference with different speeds. The obtained results are compared with FEM results and a very good agreement is observed. The advantage of the proposed semi-analytical method is high accuracy and low CPU time-consumption in case of small pre-stress deformation for realistic structures. The illustrated numerical examples can be used as benchmark for validation of numerical methods.

1. Introduction

A great deal of engineering structures have such geometry that they can be considered as shells. The mechanics of shells have been a subject of investigation for over a century. The main accomplishments are presented in number of books covering statics and/or dynamics of shells [1–6].

In case of complicated shell geometries, numerical methods are nowadays normally used. However, analytical or semi-analytical methods offer a more transparent interpretation of the results and can often serve as benchmarks for assessing the accuracy of numerical results. Analytical solutions can only be achieved for specific simplified geometries. These include for example beams, rings, plates, cylinders, spheres and tori [5]. Even with such simplified geometries, closed-form solutions are only possible and practical for certain combinations of boundary conditions.

In certain engineering situations an axisymmetric shell (shell of revolution) rotates around its axis of symmetry. This occurs, for example, with automotive tyres [7–11]. With rotating shell-like structures some interesting effects have been observed. These effects include shifts of natural frequencies due to centrifugal forces. This is because the centrifugal forces cause an initial “in-plane” membrane tension. In

* Corresponding author.
E-mail address: ivo.senjanovic@fsb.hr (I. Senjanović).

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addition, the bifurcation of natural frequencies and rotating natural modes have been observed. For example, Bryan studied vibrations of a rotating ring and described the rotating mode phenomenon [12]. Later on, many researches contributed to the field by developing the methodology for studying vibrations of rotating rings (*i.e.* [13–15]), and cylinders, (*i.e.* [16–20]). Huang and Soedel's paper on a simply supported rotating cylinder is instructive because it gives very clear mathematical and physical explanations for the phenomena of bifurcation of natural frequencies and rotating mode shapes [18].

Considering now vibrations of toroidal shells, the literature is considerably scarcer than that covering the vibration of, for example, plates or cylinders, as pointed out by Kang [21]. With toroidal shells, partial differential equations of motion can be reduced to a set of eight ordinary differential equations with variable coefficients. However, due to the variable coefficients, it is very difficult to obtain straightforward closed-form analytical solutions. In this situation, Fourier series can be used to describe the displacements. Since trigonometric functions of increasing Fourier orders are linearly independent in the range 0-2π, it is possible to obtain an infinite number of sets of eight or less ordinary differential equations with constant coefficients [22]. It is interesting to mention that a similar methodology can be used to analyse the problem of elastic stability (buckling) of a closed toroidal shell [23,24].

Energy approach is an alternative to directly solving the system of differential equation of motion. For example, Lincoln and Volterra considered free vibrations of toroidal rings theoretically and experimentally [25]. In the theoretical part of their study, they expressed the components of the elastic displacements of the ring in a Taylor's series expansion in terms of the radial and axial coordinates. The authors calculated the corresponding potential and kinetic energies of the ring and used the Hamilton's principle to determine the coefficients of the expansion. The theoretical results are also compared with results obtained experimentally on five steel toroids of different thicknesses [25].

Although in essence it is an approach based on the minimisation of energy and the variational integral, the Galerkin method has become a general algorithm for solving a variety of equations and problems [5]. For example, Leung and Kwok used the Galerkin method with complete Fourier series to describe the three displacement components of a toroidal shell segment (curved pipe), having a circular cross-section, as functions of the circumferential and meridional coordinates [26]. Ming et al. treated their curved pipe in a similar manner, however instead of using the Fourier series in both directions, they represented only the meridional mode profiles by trigonometric functions and for the circumferential mode profiles they used combinations of beam deflection functions which satisfy the boundary conditions [27]. Another application of Galerkin method has been used to investigate the effects of internal pressure on the natural frequencies of an inflated torus [28]. Some very recent works on vibration analysis of toroidal shells deal with tori made of composite layers [29], of variable thickness properties, [21], or include the effects of shear deformations and rotary inertia, [30], which are ordinarily neglected.

Considering now fully numerical procedures for the analysis of axisymmetric structures, an opened or a closed shell in the circumferential direction can be modelled by shell finite elements, [31,32]. General formulation of doubly curved shell elements is presented in [33]. For vibration analysis of a shell closed in the circumferential direction special waveguide finite elements have been developed, [34–37]. In this case a 3D problem is reduced to a 2D problem. Comparison of these two types of finite elements is presented in [38].

The present state-of-the art motivates to find a rigorous solution for the free vibrations problem of rotating and pressurised toroidal shells. The work in this paper is dedicated to this problem with a particular aim of better understanding the dynamic behaviour of rotating tires. For this purpose, the Rayleigh-Ritz method is used [39]. Ordinary strain energy, strain energy due to pre-stressing and the kinetic energy are formulated taking into account the variation of shell displacements in the circumferential direction exactly, by using simple trigonometric functions. Mode profiles of the shell cross-section (the variation in the meridional direction) are described by Fourier series. Minimizing the total energy by its differentiation per Fourier coefficients, a matrix equation of motion is obtained. The application of the presented numerical procedure is illustrated in the case of a closed toroidal shell and a thin-walled toroidal ring.

The organisation of the paper is as follows. In Section 2 general expressions of the ordinary strain and the strain energy due to the pre-stressing are formulated. Also the kinetic energy of an axisymmetric shell is derived in the co-rotating reference frame including both centrifugal and Coriolis terms. In Section 3 the analysis is narrowed down to the toroidal geometry. In Section 4 the stiffness and mass matrices are derived, and the eigenvalue problem is formulated. In Section 5 the application of the developed method is illustrated on two examples, a toroidal shell with ordinary dimensions and a thin toroidal ring. The paper also contains six appendices. In Appendices A–D variable coefficients of linear and non-linear strain energies, and submatrices of the stiffness and mass matrices are specified. Appendix E discusses lower order strain and kinetic energy terms, and Appendix F deals with the determination of tension forces due to the centrifugal load.

2. General strain – displacement relationships and energy expressions

2.1. Strain energy

Love's simplification introduced in the thin shell theory [1] enables to decouple a thin shell strain field into membrane strains due to extensional deformations and bending strains due to curvature changes

$$\tilde{\varepsilon}_{11} = \varepsilon_{11} + z\kappa_{11}, \quad \tilde{\varepsilon}_{22} = \varepsilon_{22} + z\kappa_{22} \quad \tilde{\varepsilon}_{12} = \varepsilon_{12} + z\kappa_{12}, \tag{1}$$

where z is the distance of a shell layer from the reference mid-surface. General expressions for the membrane strains and bending strains are, respectively [5]

$$\begin{aligned}
\varepsilon_{11} &= \frac{1}{A_1}\frac{\partial u_1}{\partial \alpha_1} + \frac{u_2}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{u_3}{R_1}\\
\varepsilon_{22} &= \frac{1}{A_2}\frac{\partial u_2}{\partial \alpha_2} + \frac{u_1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1} + \frac{u_3}{R_2}\\
\varepsilon_{12} &= \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{u_2}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{u_1}{A_1}\right)
\end{aligned} \tag{2}$$

$$\begin{aligned}
\kappa_{11} &= \frac{1}{A_1}\frac{\partial \beta_1}{\partial \alpha_1} + \frac{\beta_2}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\\
\kappa_{22} &= \frac{1}{A_2}\frac{\partial \beta_2}{\partial \alpha_2} + \frac{\beta_1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\\
\kappa_{12} &= \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{\beta_2}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{\beta_1}{A_1}\right),
\end{aligned} \tag{3}$$

where

$$\begin{aligned}
\beta_1 &= \frac{u_1}{R_1} - \frac{1}{A_1}\frac{\partial u_3}{\partial \alpha_1}\\
\beta_2 &= \frac{u_2}{R_2} - \frac{1}{A_2}\frac{\partial u_3}{\partial \alpha_2}
\end{aligned} \tag{4}$$

are rotation angles. The shell geometric parameters are defined in the curvilinear surface coordinate system by coordinates α_1 and α_2. Symbols A_1 and A_2 represent the two Lamé parameters, whereas R_1 and R_2 are the two radii of curvatures. u_1 and u_2 are the extensional, “in-plane”, displacements and u_3 is the “out of plane” deflection.