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Limit analysis of stochastic structures in the framework of the Probability Density Evolution Method

Kalogeris Ioannis*, Papadopoulos Vissarion

Institute of Structural Analysis and Seismic Research, National Technical University of Athens, Iroon Polytechniou 9, Zografou Campus, Athens 15780, Greece

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ABSTRACT

In the present paper a Probability Density Evolution formulation is proposed for the limit analysis of stochastic systems, which can accurately and efficiently evaluate the effect the system's random parameters have on its nonlinear and limit response. The proposed formulation of the classic Probability Density Evolution Method reduces the corresponding Generalized Density Evolution equations, which are partial differential equations, to a system of ordinary differential equations, that can be efficiently solved with the method of characteristics. With this reformulation, the cumulative distribution function of the critical load of the structure can be accurately and efficiently evaluated. The estimation of stochastic limit buckling loads of imperfection sensitive structures is a natural extension of this method. In addition, a methodology is put forward for the estimation of the probabilistic characteristics of the full load-displacement curve for a stochastic nonlinear system in the context of Newton-Raphson incremental-iterative schemes. The main advantage of the proposed approaches is that they allow for a quantification of the effect of uncertainties on the structural capacity, with only a small number of deterministic analyses compared to Monte Carlo simulation. The applicability and validity of the proposed methodology for limit and nonlinear structural analysis is verified through extensive numerical investigations.

1. Introduction

In an effort to achieve more reliable representations in real-life engineering problems, researchers quickly realized that this could not be accomplished without taking into account the role of uncertainties in the system under consideration. This notion led to the development of the stochastic finite element method (SFEM) and in the past few years significant advances have been accomplished in its application to structural engineering problems. The most eminent and widely used methods developed to treat randomness in such problems could be broadly classified into the following categories: (a) Monte Carlo simulation techniques [1] and its variants (importance sampling [2], subset simulation [3], line sampling [4]), (b) random perturbation techniques [5], (c) the spectral stochastic finite element method [6] and (d) path integral techniques [7,8]. Despite the progress being made, the application of these methods in large scale realistic problems is still limited, either due to prohibitive computational costs or due to their inability to deal with structures with complex behavior. Recently, a new approach was added to the family of stochastic methods, that is, the Probability Density Evolution Method (PDEM) [9]. This method possesses the wide range of applicability of the Monte Carlo methods, whereas, in many cases, requires a considerably smaller number of deterministic analyses. Although mainly employed in dynamic systems [10,11], a reformulation of the method was proposed in [12] to also treat static systems.

On the other hand, a valuable tool in structural engineering for determining the capacity of a structure beyond its elastic region is the well-known limit analysis. The two main approaches when one wants to perform a limit analysis, are the mathematical programming based approaches [13–16] and the displacement based direct stiffness method, which is utilized in most commercial programs. Nevertheless, in practice it quickly became evident that, even with the most elaborate and sophisticated deterministic models, the results between the computer analyses and the experiments did not fully agree [17,18]. The reason for this lies in the key role, that imperfections in geometry, material and section properties [19] play in the actual behaviour of the structure. At this point, the role of random parameters in a structural system and their influence on its inelastic collapse limit load becomes evident and this fact necessitates the extension of limit analysis to stochastic systems for more realistic and accurate models, especially in the framework of system reliability analysis. Towards this direction, many approaches can be found in the literature including the direct Monte Carlo simulation [20], the β-unzipping method [21], bounding techniques [22] and analytical stochastic response analysis [23]. In this framework, stochastic buckling analysis has also received great

* Corresponding author.
E-mail addresses: yianniskalogeris@gmail.com (I. Kalogeris), vpapado@central.ntua.gr (V. Papadopoulos).

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attention. The existence of random imperfections in almost every real structure leads to a wide scattering of the buckling loads and the approaches to estimate their variability include either the assumption of imperfections with variable amplitude in the form of the critical eigenmode for the perfect structure [24,25], or the more sophisticated representation of the imperfections as random fields [26–29]. In [30], random fields were used to represent the initial out-of-straightness of steel columns and evaluate the stochastic buckling trajectories.

In the present paper a Probability Density Evolution formulation is proposed for the limit analysis of stochastic systems, which can accurately and efficiently evaluate the effect the system's random parameters have on its nonlinear and limit response. The proposed formulation of the classic Probability Density Evolution Method reduces the corresponding Generalized Density Evolution equations (GDEEs), which are partial differential equations, to a system of ordinary differential equations, that can be efficiently solved with the method of characteristics [31,32]. With this reformulation, the cumulative distribution function of the critical load of the structure can be accurately and efficiently evaluated. The estimation of stochastic limit buckling loads of imperfection sensitive structures is a natural extension of this method. In addition, a methodology is put forward for the estimation of the probabilistic characteristics of the full load-displacement curve for a stochastic nonlinear system in the context of Newton-Raphson incremental-iterative schemes. The main advantage of the proposed approaches is that they allow for a quantification of the effect of uncertainties on the structural capacity, with only a small number of deterministic analyses compared to a Monte Carlo simulation. The applicability and validity of the proposed methodology for limit and nonlinear structural analysis is verified through extensive numerical investigations.

The remaining of this paper is organized as follows: In Section 2, the formulation of the PDEM in static cases is outlined. In Section 3, the partitioning of the probability-assigned space, as well as the Karhunen-Loève expansion for random fields are described. Section 4 presents the implementation aspects of applying the PDEM in the estimation of stochastic limit loads. In Section 5, an approach is put forward that enables the coupling between PDEM and Newton-Raphson solution algorithms. Lastly, in Section 6, the proposed methodology is applied in test cases and the results obtained are critically assessed and discussed further.

2. Probability Density Evolution Method for static problems

PDEM was introduced by Li and Chen [9] and offers a relatively new approach in solving the stochastic conservation equations, that govern the flow of probability in a stochastic system. Such a solution would give the complete probabilistic information about the system under consideration rather than the second order statistics. This method takes advantage of the principle of preservation of probability, but unlike the other probability density evolution equations (Liouville equation, Fokker-Planck equation), it is viewed from the random event perspective, leading to a family of GDEEs, which are numerically tractable.

These concepts were applied in [12], where a reformulation of the method was proposed for stochastic static system. According to this reformulation the equation of equilibrium of a multi-degree-of freedom system with random system parameters is considered as

$$\mathbf{K}(\theta)\cdot\mathbf{u} = \mathbf{F}(\theta) \tag{1}$$

where $\mathbf{K}$ is the system's stiffness matrix, $\mathbf{u}$ stands for the displacement vector and $\mathbf{F}$ is the external force vector; θ is the vector of all random parameters involved in the physical properties of the system.

In Fig. 1, a generic element is depicted, where the loading q and the bending stiffness EI are assumed random.

As a consequence, the displacement u at each position x of the element will also be random. Moreover, the parameter θ can also be seen as the stochastic input for this system and since no other source of

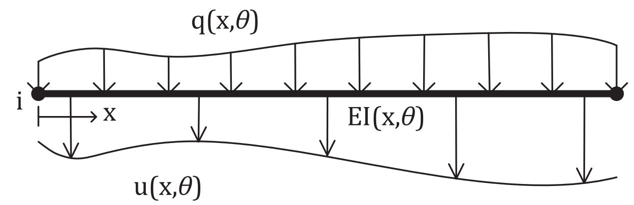

Fig. 1. Generic element with the randomness involved in the loading and the system parameters.

randomness enters or exits the system along the position x, then the probability is a conserved quantity. This property is mathematically expressed in the following equation

$$\frac{D}{Dx}\int_{\Omega_x\times\Omega_\theta} p_{u\theta}(u,\theta,x)\mathrm{d}u\mathrm{d}\theta = 0 \tag{2}$$

where $\frac{D}{Dx}$ is the material derivative with respect to the position x,$\Omega_x \times \Omega_\theta$ is the product space of the physical and random space Ω_x and Ω_θ, respectively, and $p_{u\theta}$ denotes the joint probability density function (pdf) of $(u(x),\theta)$. After some mathematical manipulations, which shall be omitted here for the sake of brevity (the reader is referred to [33] for more details), Eq. (2) can be recast for any arbitrary $\Omega_\theta \subseteq \Omega_\theta$ in the following form

$$\int_{\Omega_\theta}\left(\frac{\partial p_{u\theta}(u,\theta,x)}{\partial x} + \frac{du(\theta,x)}{dx}\frac{\partial p_{u\theta}(u,\theta,x)}{\partial u}\right)\mathrm{d}\theta = 0 \tag{3}$$

By exploiting further the fact that Eq. (3) holds for any $\Omega_\theta \subseteq \Omega_\theta$, we could partition Ω_θ into subdomains Ω_q' $s,q = 1,2,\dots,n_{pt}$, such that $\Omega_i \cap \Omega_j = \varnothing$,$\forall\ i \neq j$ and $\cup_{q=1}^{n_{pt}}\Omega_q = \Omega_\theta$. Then, Eq. (3) is rewritten as

$$\int_{\Omega_q}\left(\frac{\partial p_{u\theta}(u,\theta,x)}{\partial x} + \frac{du(\theta,x)}{dx}\frac{\partial p_{u\theta}(u,\theta,x)}{\partial u}\right)\mathrm{d}\theta = 0, \quad q = 1,2,\dots,n_{pt} \tag{4}$$

and since Eq. (4) is valid for any Ω_q, and by making the assumption that

$$\frac{du(\theta,x)}{dx} = \frac{du(\theta_q,x)}{dx} \quad \text{for } \theta \in \Omega_q \tag{5}$$

then,

$$\frac{\partial p_{u\theta}(u,\theta_q,x)}{\partial x} + \frac{du(\theta_q,x)}{dx}\frac{\partial p_{u\theta}(u,\theta_q,x)}{\partial u} = 0, \quad q = 1,2,\dots,n_{pt} \tag{6}$$

Eqs. (6) are the well-known GDEEs. In order to solve these pde's, the initial conditions for the problem are required, which usually are of the form

$$p_{u\theta}(u,\theta,x)|_{x=x_0} = \delta(u-u_0)\overline{p}_q \tag{7}$$

where

$$\overline{p}_q = \int_{\Omega_q} p_\theta(\theta)d\theta \tag{8}$$

These initial conditions can be found at positions, where there exist boundary conditions in the physical problem. For instance, if u at $x = x_0$ is fixed, then $u(x_0,\theta_q) = 0$ for all θ_q with $q = 1,\dots,n_{pt}$. In terms of probability, this observation translates to the fact that, all probability there should be concentrated at the event of zero displacement. Also, $\frac{du(\theta_q,x)}{dx}$ in Eq. (6) is the flux term, which can be evaluated directly, by solving the governing Eq. (1) for the random event θ_q.

Next, if we denote

$$p_q(u,x) = \int_{\Omega_q} p_{u\theta}(u,\theta,x)\mathrm{d}\theta, \quad q = 1,2,\dots,n_{pt} \tag{9}$$

then, the solution is

$$p(u,x) = \sum_{q=1}^{n_{pt}} p_q(u,x) \tag{10}$$

The numerical procedure to solve (2) is summarized in the following steps: