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An effective method for the evaluation of the pdf response of dynamic systems subjected to non-stationary loads

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ABSTRACT

This paper deals with the evaluation of the response probability density function (pdf) and/or characteristic function (cf) of linear systems subject to stochastic loads. It proposes a new method based on the new version of Probabilistic Transformation Method (PTM). An important aspect of the proposed approach is the ability to join directly the pdf of the input load with that of the response. Working in term of pdfs and/or cfs, the stochastic properties of the response process are full described. This enables to take into account the non Gaussianity and non stationarity of the process. Based on the step-by-step integration method, explicit solutions will be proposed for the random response of systems driven by loads stochastically defined by their cf or loads defined by a quadratic relation of a Gaussian process.

1. Introduction

The characterization of the random response of a structural time-dependent system often requires a high computational effort, which in turns depends on the type of random loads. If the load is modelled as a stationary Gaussian random process, the quantities of interest able to describe the random properties of the response are the correlations and the cross-correlations, in the time domain, and the power spectral densities, in the frequency domain. In the case of stationary loads, the analysis procedures are largely reported in the literature [1–3]. The case of non-stationary excitations is more complicated, and some efforts have been made by several authors to propose effective methods [4–11]. Moreover, the effort increases if the load is non Gaussian because the second order correlations and the cross-correlations of the response are not sufficient to describe the random properties. In this case, the full probabilistic characterization of the random response is given by the knowledge of its probability density function (pdf), or by its characteristic function (cf). Unfortunately there are not exact solutions, except in some simple cases, such as for linear systems subject to Gaussian inputs.

The literature shows several methods that allow reconstructing the pdf response by using the moments (or cumulants) series method, with a relatively low computational effort [12–16]. The validity of these approaches is largely confirmed; however they lack a direct nature, namely the ability to join directly the pdf of the input with that of the output. Also, in the case of strongly non Gaussian processes, a very high number of moments/cumulants is necessary and the convergence of these methods is not guaranteed; and, at last, a very high computational effort is usually related to them. Monte Carlo methods [17,18] exhibit the well known problem that the accuracy of the estimates depends on the sampling size of the stochastic processes, besides of the number of samples, increasing the related computational effort. Even these methods, moreover, do not define a direct input–output relationship in terms of pdf. Nevertheless, advances of probability density evolution method have been obtained, based on the principle of preservation of probability [19,20]. The key point of this approach is that the probability density evolution of response of structural systems relies upon their physical mechanism.

Aim of this paper is showing a method, based on the Probabilistic Transformation Method (PTM) [21–26], to obtain the pdf response of linear dynamic systems subjected to non-stationary and non-Gaussian input processes. This approach enables to evaluate the response pdf and/or cf once that the input pdf and/or cf is known. This allows considering both the non-stationarity and non-Gaussianity of the process directly by assigning the input pdf/cf law. This method makes to obtain the pdf response with a very low computational effort. Besides, some explicit relationships are proposed for the response cf.

2. Preliminary concepts

The differential equation governing the dynamic behaviour of a multi-degrees-of-freedom linear system is usually written as follows:

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$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{A}\mathbf{f}(t) \tag{1}$$

where **M**, **C** and **K** are the system mass, damping and stiffness $n \times n$ matrices, respectively; $\mathbf{u}(t)$ is the n-vector that collects the degrees of freedom; **A** is a $n \times m$ matrix of coefficients and $\mathbf{f}(t)$ is the m-vector of the external loads. In the state variable space, the equation of motion is rewritten in the following form:

$$\dot{\mathbf{x}}(t) = \mathbf{D}\mathbf{x}(t) + \mathbf{v}\mathbf{f}(t) \tag{2}$$

where

$$\mathbf{x}(t) = \begin{pmatrix} \mathbf{u}(t) \\ \dot{\mathbf{u}}(t) \end{pmatrix}; \quad \mathbf{D} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{pmatrix}; \quad \mathbf{v} = \begin{pmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{A} \end{pmatrix} \tag{3a-c}$$

The vector $\mathbf{x}(t)$, which collects the response state variables, can be evaluated by the Duhamel's integral, that is:

$$\mathbf{x}(t) = \mathbf{\Theta}(t)\mathbf{x}_0 + \int_0^t \mathbf{\Theta}(t-\tau)\mathbf{v}\mathbf{f}(\tau)d\tau \tag{4}$$

where $\mathbf{x}_0$ is the vector collecting the initial conditions at $t = 0$; $\mathbf{\Theta}(t)$ is the fundamental matrix related to the differential equation of motion, that can be defined in the following way:

$$\mathbf{\Theta}(t) = \exp(\mathbf{D}t) \tag{5}$$

The response system $\mathbf{x}(t)$ can be evaluated numerically by several methods; among these, the step-by step integration method [2,27] based on the fundamental matrix, will be used in this work. This method enables us to solve in closed form, in a generic step Δt, the convolution integral given into Eq. (4) once that a polynomial interpolation law is assumed for the vector load $\mathbf{f}(t)$ in correspondence of the same time step Δt. As an example, assuming for $\mathbf{f}(t)$ a linear interpolation law within the interval $[t_{k-1}, t_k]$, one obtains the following step-by-step numerical procedure:

$$\mathbf{x}(t_k) = \mathbf{x}(k\Delta t) = \mathbf{\Theta}(\Delta t)\mathbf{x}(t_{k-1}) + \mathbf{\Gamma}_0(\Delta t)\mathbf{f}(t_{k-1}) + \mathbf{\Gamma}_1(\Delta t)\mathbf{f}(t_k) \tag{6}$$

$\mathbf{x}(t_k)$ being the system response at the time $t_k = k\Delta t$; analogously $\mathbf{f}(t_k)$ is the vector load evaluated at the same time. The operators $\mathbf{\Gamma}_0(\Delta t)$ and $\mathbf{\Gamma}_1(\Delta t)$ are given by the following relationships:

$$\mathbf{\Gamma}_1(\Delta t) = \left[\frac{\mathbf{L}(\Delta t)}{\Delta t} - \mathbf{I}\right]\mathbf{D}^{-1}\mathbf{v}; \quad \mathbf{\Gamma}_0(\Delta t) = \left[\mathbf{\Theta}(\Delta t) - \frac{\mathbf{L}(\Delta t)}{\Delta t}\right]\mathbf{D}^{-1}\mathbf{v}; \\ \mathbf{L}(\Delta t) = [\mathbf{\Theta}(\Delta t) - \mathbf{I}]\mathbf{D}^{-1} \tag{7a-c}$$

Without losing the generality of the approach, deterministic zero initial conditions are considered. Appling recursively the step-by-step procedure given above in Eq. (6), it s possible to define the relationship between the response system at the time t_k and all the vectors load $\mathbf{f}(t_i)$, $i = 0,1, \ldots, k$, that is:

$$\begin{aligned} \mathbf{x}(t_k) &= \mathbf{x}(k\Delta t) \\ &= \mathbf{\Theta}^{k-1}(\Delta t)\mathbf{\Gamma}_0(\Delta t)\mathbf{f}(t_0) \\ &\quad + \sum_{i=1}^{k-1}\left[\mathbf{\Theta}^{k-i}(\Delta t)\mathbf{\Gamma}_1(\Delta t) + \mathbf{\Theta}^{k-i-1}(\Delta t)\mathbf{\Gamma}_0(\Delta t)\right]\mathbf{f}(t_i) + \mathbf{\Gamma}_1(\Delta t)\mathbf{f}(t_k) \end{aligned} \tag{8}$$

This equation can be rewritten in compact form as follow:

$$\mathbf{x}(t_k) = \mathbf{R}^{(k)}\mathbf{f}^{(k)}; \\ \mathbf{R}^{(k)} = \left[\mathbf{r}_0^{(k)}, \mathbf{r}_1^{(k)}, \ldots, \mathbf{r}_k^{(k)}\right]; \quad \mathbf{f}^{(k)} = \left[\mathbf{f}^T(t_0), \mathbf{f}^T(t_1), \ldots, \mathbf{f}^T(t_k)\right]^T \tag{9a-c}$$

where

$$\mathbf{r}_i^{(k)} = \mathbf{\Theta}^{k-i}(\Delta t)\mathbf{\Gamma}_1(\Delta t) + \mathbf{\Theta}^{k-i-1}(\Delta t)\mathbf{\Gamma}_0(\Delta t) = \mathbf{\Theta}\mathbf{r}_i^{(k-1)}; \\ \text{with } i = 1, 2, \ldots, k-1 \\ \mathbf{r}_0^{(k)} = \mathbf{\Theta}^{k-1}(\Delta t)\mathbf{\Gamma}_0(\Delta t) = \mathbf{\Theta}\mathbf{r}_0^{(k-1)}; \quad \mathbf{r}_k^{(k)} = \mathbf{\Gamma}_1(\Delta t) \tag{10a-c}$$

Eq. (9a) provides the dynamic response at the time t_k explicitly respect to the random load $\mathbf{f}(t)$, which is sampled, and all the samples are collected in the vector $\mathbf{f}^{(k)}$. By this procedure, the dynamic problem can be addressed like a static one. All the vectors load $\mathbf{f}(t_i)$, with $i = 0, 1, \ldots, k$, appearing in Eq. (9c), can be considered as samples of the stochastic process $\mathbf{f}(t)$ extracted at the sampling step $t_j = j\Delta t$; correspondently, the response vector $\mathbf{x}(t_k)$ is the sample response of the stochastic process $\mathbf{x}(t)$ at the time t_k. In order to characterize probabilistically the stochastic process $\mathbf{x}(t)$, one could characterize the sample response $\mathbf{x}(t_k)$: this is what made in this work and, as it will be shown later, the fundamental approach to obtain this result is the PTM, whose basic concepts are shown in next section.

3. Basic concept of the PTM

The PTM [21–26,28–31] is based on some relationships that enable to join the joint probability density functions (jpdfs) of two random vectors connected by a deterministic law. However, in this work it will focus only on the integral relationship showed below. Let's consider a n-dimensional random vector **x** and a m-dimensional application $\mathbf{h}(\cdot)$ such that:

$$\mathbf{x} = \mathbf{h}(\mathbf{f}) \tag{11}$$

x being a random vector, as well as **f**. It can be shown that the jpdfs of **x** and **f**, that are $p_{\mathbf{x}}(\mathbf{x})$ and $p_{\mathbf{f}}(\mathbf{f})$, are related by the following relationship [21–26]:

$$p_{\mathbf{x}}(\mathbf{x}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} ..(m).. \int_{-\infty}^{+\infty} p_{\mathbf{f}}(\mathbf{y})\delta(\mathbf{x} - \mathbf{h}(\mathbf{y}))d\mathbf{y} \tag{12}$$

$\delta(\mathbf{x} - \mathbf{h}(\mathbf{y}))$ being the m-dimensional Dirac Delta centred in the coordinate vector $\mathbf{h}(\mathbf{y})$, that is:

$$\delta(\mathbf{x} - \mathbf{h}(\mathbf{y})) = \delta(x_1 - h_1(\mathbf{y}))\delta(x_2 - h_2(\mathbf{y}))\ldots\delta(x_m - h_m(\mathbf{y})) \tag{13}$$

where $h_j(\cdot)$ (with $j = 1, 2 \ldots, m$) is the jth element of the m-dimensional application $\mathbf{h}(\cdot)$.

Eq. (12) provides a direct relation between the jpdfs of the random vectors **x** and **f** by a multidimensional integral. From Eq. (12) it is possible to obtain the integral relationship of every marginal pdf $p_{x_j}(x_j)$ by integrating respect to all the other variables and taking into account the properties of the Dirac Delta function. For example, the marginal pdfs and the joint second order pdfs have the following integral form:

$$p_{x_j}(x_j) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} ..(m).. \int_{-\infty}^{+\infty} p_{\mathbf{f}}(\mathbf{y})\delta(x_j - h_j(\mathbf{y}))d\mathbf{y} \\ p_{x_j x_p}(x_j, x_p) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} ..(m).. \int_{-\infty}^{+\infty} p_{\mathbf{f}}(\mathbf{y})\delta(x_j - h_j(\mathbf{y}))\delta(x_p - h_p(\mathbf{y}))d\mathbf{y} \tag{14a,b}$$

From Eq. (14) it is possible to obtain the integral relationships of the characteristic functions; by applying the Fourier transform to both sides of Eq. (14) one obtains:

$$\begin{aligned} M_{x_j}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} p_{x_j}(x_j)\exp(-i\omega x_j)dx_j \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} ..(m).. \int_{-\infty}^{+\infty} p_{\mathbf{f}}(\mathbf{y})\exp(-i\omega h_j(\mathbf{y}))d\mathbf{y} \\ M_{x_j x_p}(\omega_j, \omega_p) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p_{x_j x_k}(x_j, x_p)\exp(-i\omega_j x_j - i\omega_p x_p)dx_j dx_p \\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} ..(m).. \int_{-\infty}^{+\infty} p_{\mathbf{f}}(\mathbf{y})\exp(-i\omega_j h_j(\mathbf{y}) - i\omega_p h_p(\mathbf{y}))d\mathbf{y} \end{aligned} \tag{15a,b}$$

Eqs. (14)–(15) are the reference relationships of the new version of the PTM proposed in [21–26] for systems driven by static loads. In particular, Eq. (15) is the reference relationship of this work. Now,