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Bending and buckling of tapered steel beam structures

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ABSTRACT

This paper describes an efficient finite element method of analysing the elastic in-plane bending and out-of-plane buckling of indeterminate beam structures whose members may be tapered and of mono-symmetric I cross-section. The structure's loading includes concentrated moments and concentrated or uniformly distributed off-axis transverse and longitudinal forces, and its deformations may be prevented or resisted by concentrated or continuous rigid or elastic off-axis restraints.

Tapered finite element formulations are developed by numerical integration instead of the closed forms often used for uniform elements. Difficulties in specifying the load positions for tapered mono-symmetric members caused by the variations of the centroidal and shear centre axes are avoided by using an arbitrary axis system based on the web mid-line. Account is taken of additional Wagner torque terms arising from the inclination of the shear centre axis.

A computer program based on this method is used to analyse a number of examples of the elastic in-plane bending of tapered cantilevers and built-in beams, and very close agreement is found between its predictions and closed form solutions.

The program's predictions of the elastic out-of-plane flexural–torsional buckling of a large number of uniform and tapered doubly and mono-symmetric beams and cantilevers under various loading and restraint conditions are generally in close agreement with existing predictions and test results. The common approximation in which tapered elements are replaced by uniform elements is shown to converge slowly, and to lead to incorrect predictions for tapered mono-symmetric beams.

1. Introduction

The distributions of axial forces and bending moments in steel beam structures caused by the applied loads need to be determined before their elastic flexural–torsional buckling resistances can be analysed [1]. While these may be easily found for indeterminate structures with uniform members and for determinate structures with tapered members, they are less easily found for indeterminate tapered structures. It is noteworthy that most if not all of the early and later studies of the elastic buckling of tapered structures cited in [2,3] and more recently in [4,5] are for statically determinate beams or cantilevers.

The finite element stiffness matrices used to analyse the elastic in-plane bending and out-of-plane buckling of a tapered beam structure depend on the variations of the section properties along each element of which a structure is composed, and in general require numerical integrations to be made, in place of the use of the common closed form solutions for uniform elements.

When the tapered elements are of mono-symmetric cross-section (Fig. 1a), the centroidal and shear centre axes wander [6] along the element length (Fig. 1b). If the problem is simplified by using an arbitrary axis system [7], then modifications need to be made for the eccentricity of the centroid from the chosen axis to allow for the moments induced by eccentric longitudinal loads or restraints.

This paper describes a finite element method of analysing the elastic in-plane bending and out-of-plane buckling of indeterminate beam structures whose members may be tapered and of mono-symmetric I-section. The structure's loading includes concentrated moments and concentrated or uniformly distributed off-axis transverse and longitudinal forces, and its deformations may be prevented or resisted by concentrated or continuous rigid or elastic off-axis restraints. The method is superior to the commonly used approximate method of replacing a tapered element by a large number of uniform elements, with a significant reduction in the number of elements required to obtain an accurate solution.

While this method gives solutions for bending and out-of-plane buckling, it also allows the analysis of the inelastic buckling of uniform steel beams for which non-uniform yielding along the beam causes both the in-plane and out-of-plane properties to vary, so that the member is effectively tapered [8]. It may be noted that the combination of the anti-symmetrical stresses due to the

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Nomenclature

A	area of cross-section
$b_{b,t,w}$	bottom and top flange widths and web depth
$D_{i,u,v,ut}$	generalised elasticity matrices
E	Young's modulus of elasticity
f	stress
G	shear modulus of elasticity
G_o	global stability matrix
g_{oe}	element stability matrix
$I_{x,y}$	second moments of area about x, y axes
$I_{b,t}$	second moments of area of bottom and top flanges
I_0	value of I_x at $z = 0$
I_p	$I_x + I_y$
I_w	warping section constant
J	uniform torsion constant
$K_{i,o}$	global in-plane and out-of-plane stiffness matrices
$k_{ie,oe}$	element in-plane and out-of-plane stiffness matrices
L	length
M	external moment
M_x	internal bending moment
N, n	concentrated and distributed longitudinal loads
N_z	internal compression force
Q, q	concentrated and distributed transverse loads
$Q_{ie,in}$	generalised equivalent distributed and concentrated nodal loads
Q_u	value of Q for a uniform member
r_s	distance of point (x, y) from shear centre (0, y_s)
$T_{ie,oe}$	in-plane and out-of-plane element to global transformation matrices
T_w	Wagner torque
$t_{b,t,w}$	Bottom and top flange and web thicknesses
U_e	Element strain energy increase
u, v, w	Displacements in x, y, z directions
u_s	Shear centre displacement parallel to x axis
V_e	potential energy increase
x, y	cross-section coordinates of arbitrary axis system
$y_{c,s}$	centroid and shear centre coordinates
y_o	distance of shear centre from centroid
$y_{N,n}$	positions of N, n loads
$y_{Q,q}$	positions of Q, q loads
$y_{R,r}$	positions of concentrated and distributed restraints
z	distance along member
$\alpha_{d,t,w}$	taper constants
β	$=(1 + \alpha z/L)$
β_x	Mono-symmetry section constant
$\Delta_{i,o}$	global in-plane and out-of-plane nodal deflections and rotations
$\delta_{ie,oe}$	element in-plane and out-of-plane nodal deflections and rotations
$\varepsilon_{u,v}$	generalised strain vectors
λ	buckling load factor
ϕ	twist rotation

applied loading with the symmetrical residual stresses in a doubly symmetric I-section beam causes the effective section to become mono-symmetric [9].

A computer program FTBTM based on the method described in this paper is used to analyse a number of examples of the bending and out-of-plane buckling of tapered beam structures, and its predictions are compared with previous theoretical and experimental values.

2. Tapered monosymmetric members

The member cross-section shown in Fig. 1a has top and bottom flange and web widths b_t, b_b, and b_w and thicknesses t_t, t_b, and t_w, respectively. The element shown in Fig. 1b may be linearly tapered in all cross-section dimensions, and while thickness tapering is very rare in practice, depth tapering is comparatively common. The member has an arbitrary but convenient [7] longitudinal axis Oz which is the locus of the web mid-heights, and an Oy axis which coincides with the web centreline. The section properties required for an in-plane analysis are

$$I_0 = A = \int_A dA = b_t t_t + b_b t_b + b_w t_w \quad (1)$$

$$I_1 = y_c A = \int_A y dA = -b_t t_t b_w/2 + b_b t_b b_w/2 \quad (2)$$

$$I_2 = I_x - y_c^2 A = \int_A y^2 dA$$
$$= b_t t_t b_w^2/4 + b_b t_b b_w^2/4 + b_w^3 t_w/12 + b_w t_w y_c^2 \quad (3)$$

in the coordinate y_c of the centroid is given by

$$y_c = \frac{b_w}{2}\frac{b_b t_b - b_t t_t}{A} \quad (4)$$

The coordinate of the shear centre S (Fig. 1a) is defined by

$$y_s = y_c + y_0 = \frac{b_w}{2}\frac{b_b^3 t_b - b_t^3 t_t}{b_b^3 t_b + b_t^3 t_t} \quad (5)$$

Concentrated transverse and longitudinal loads Q (at y_Q), N (at y_N), and moments M may act at a point z along the axis Oz as shown in Fig. 2a, as well as rigid or elastic restraints (at y_R). Distributed transverse and longitudinal loads q (at y_q) and n (at y_n) may act along a length L of the element as shown in Fig. 2b, as well as elastic restraints (at y_r).

3. In-plane analysis and behaviour

3.1. Finite element formulation

A finite element method of carrying out the in-plane (pre-flexural–torsional buckling) linear elastic analysis of uniform beam-columns is detailed in [1]. In this, the equilibrium equations are represented by

$$K_i \Delta_i = Q_{in} + Q_{ie} \quad (6)$$

in which K_i is the in-plane global stiffness matrix, Δ_i are the in-plane global nodal deflections and rotations, Q_{in} are the nodal loads, and Q_{ie} are the nodal loads equivalent to the loads distributed along the elements. The global stiffness matrix is given by

$$K_i = \sum_e T_{ie}^T k_{ie} T_{ie} \quad (7)$$

in which k_{ie} is an element stiffness matrix and T_{ie} is a matrix which transforms the element in-plane end deflections and rotations δ_{ie} into the corresponding global deflections and rotations Δ_i according to

$$\delta_{ie} = T_{ie} \Delta_i \quad (8)$$